

# Broken Weyl-Invariance and the Origin of Mass

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## Abstract

A massless Weyl-invariant dynamics of a scalar, a Dirac spinor, and electromagnetic fields is formulated in a Weyl space,  $W_4$ , allowing for conformal rescalings of the metric and of all fields with nontrivial Weyl weight together with the associated transformations of the Weyl vector fields  $\kappa_\mu$  representing the  $D(1)$  gauge fields with  $D(1)$  denoting the dilatation group. To study the appearance of nonzero masses in the theory the Weyl-symmetry is broken explicitly and the corresponding reduction of the Weyl space  $W_4$  to a pseudo-Riemannian space  $V_4$  is investigated assuming the breaking to be determined by an expression involving the curvature scalar  $R$  of the  $W_4$  and the mass of the scalar, selfinteracting field. Thereby also the spinor field acquires a mass proportional to the modulus  $\Phi$  of the scalar field in a Higgs-type mechanism formulated here in a Weyl-geometric setting with  $\Phi$  providing a potential for the Weyl vector fields  $\kappa_\mu$ . After the Weyl-symmetry breaking one obtains generally covariant and  $U(1)$  gauge covariant field equations coupled to the metric of the underlying  $V_4$ . This metric is determined by Einstein's equations, with a gravitational coupling constant depending on  $\Phi$ , coupled to the energy-momentum tensors of the now massive fields involved together with the (massless) radiation fields.

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## I. INTRODUCTION

In order to understand the origin of nonzero masses in physics it is suggestive to start from a theory, having no intrinsic scales, which is formulated in a Weyl space, i.e. in a space-time  $W_4$  being equivalent to a family of Riemannian spaces

$$(g_{\mu\nu}, \kappa_\rho), (g'_{\mu\nu}, \kappa'_\rho), (g''_{\mu\nu}, \kappa''_\rho) \dots \quad (1.1)$$

characterized by a class of metric tensors,  $g_{\mu\nu}(x)$ , and Weyl vector fields,  $\kappa_\rho(x)$ , related by

$$g'_{\mu\nu}(x) = \sigma(x) g_{\mu\nu}(x), \quad (1.2)$$

$$\kappa'_\rho(x) = \kappa_\rho(x) + \partial_\rho \log \sigma(x), \quad (1.3)$$

where  $\sigma(x) \in D(1)$ , with the dilatation group  $D(1)$  being isomorphic to  $R^+$  (the positive real line). The transformations (1.2) and (1.3) are called *Weyl transformations* involving a conformal rescaling of the metric given by (1.2) together with the transformation (1.3) of the Weyl vector field. A Weyl space is thus characterized by two fundamental forms:

$$ds^2 = g_{\mu\nu}(x) dx^\mu \otimes dx^\nu; \quad \kappa = \kappa_\mu(x) dx^\mu, \quad (1.4)$$

with the quadratic differential form  $ds^2$  describing the distances on the space-time manifold and with the linear differential form  $\kappa$  determining the local change of the unit of length used for measuring the distances. (Compare Weyl's original proposal of 1918 [1] as well as his book *Raum-Zeit-Materie* [2]. For a general discussion of Weyl spaces and conformal symmetry in physics see Fulton, Rohrlich and Witten [3]. Of course, we shall here not identify the Weyl vector fields  $\kappa_\mu$  with the electromagnetic potentials  $A_\mu$  as Weyl originally did.) In the following we shall use a Weyl space of dimension  $d = 4$  with Lorentzian signature  $(+,-,-,-)$  of its metric. We collect the relevant formulae characterizing the geometry of a Weyl space in Appendix A – C.

A  $W_4$  reduces to a pseudo-Riemannian space  $V_4$  for  $\kappa_\mu = 0$ ; a  $W_4$  is equivalent to a  $V_4$  if the “length curvature” (Weyl's “Streckenkrümmung”) associated with the Weyl vector fields  $\kappa_\mu$  is zero, i.e.

$$f = d\kappa = 0, \quad (1.5)$$

and the transfer of the unit of length is integrable.

We shall use in this paper a Weyl geometry as a geometric stratum for the formulation of a physical theory involving, at the beginning, only massless fields. The theory presented in Sect. II is characterized by a Weyl-invariant Lagrangean density,  $\mathcal{L}_{W_4}$ , involving besides the fields  $g_{\mu\nu}$  and  $\kappa_\rho$  a complex massless scalar field  $\varphi$  as well as a massless Dirac spinor field  $\psi$ . The local group structure of the theory is given by the gauge group  $SO(3, 1) \otimes D(1)$  – or rather by  $Spin(3, 1) \otimes D(1)$  for the Dirac spinors. (Compare the definition of the relevant frame bundles given in Appendix A and in Sect. II below.) In Sect. III we extend the description by including electromagnetism ( $A = A_\mu(x)dx^\mu$ ;  $F = dA$ ) using the minimal electromagnetic coupling procedure yielding a Lagrangean density,  $\tilde{\mathcal{L}}_{W_4}$ , for a massless theory with gauge group  $SO(3, 1) \otimes D(1) \otimes U(1) \simeq SO(3, 1) \otimes Gl(1, \mathbb{C})$ , or its covering group  $Spin(3, 1) \otimes D(1) \otimes U(1)$  for the Dirac spinor fields. In Sect. IV this  $U(1)$  and  $D(1)$  gauge covariant as well as generally covariant – i.e. Weyl-covariant – theory involving “gravitation” which describes a Weyl-symmetric dynamics for the metric in the form of field equations of Einstein type, is broken to a theory defined on a space-time  $V_4$  by the introduction of a Weyl-symmetry breaking term depending on the scalar curvature invariant of the  $W_4$  giving thereby a nonzero mass to the scalar field  $\varphi$ . This reduction  $W_4 \rightarrow V_4$  is governed by a Weyl vector field of the type  $\kappa_\rho = \partial_\rho \chi$  being “pure gauge”, i.e. being given by a gradient of a scalar function yielding thus zero length curvature:  $f = 0$ . At the same time also the Dirac field acquires a mass due to a Yukawa-type coupling between the  $\varphi$  and the  $\psi$  fields allowed by Weyl-invariance and the adopted Weyl weights of the fields involved (see below). An essential part in the Weyl-symmetry breaking and the concomitant mass generation is the requirement that Einstein’s equations for the metric are obtained in the limiting  $V_4$  description in coupling ultimately the purely metric geometry to the energy-momentum tensors of the massive fields  $\varphi$  and  $\psi$  as well as to the radiation field  $F$ . Our aim in this paper is thus to study a Higgs-type phenomenon in a geometric setting containing gravitation from

the outset – or, more specifically – to investigate such a mass giving phenomenon as the breaking of a Weyl-symmetry yielding a Riemannian description in the limit.

The work presented in this paper is an extension of an investigation of a theory involving only a complex scalar quantum field,  $\varphi$ , which was carried out by one of us [4]. It turned out in this investigation that the  $\varphi$ -field did not behave as a bona fide matter field in this Weyl geometric framework. When split into a modulus  $\Phi(x)$  and a phase  $S(x)$ , i.e.

$$\varphi = \Phi e^{\frac{i}{\hbar}S}; \quad \Phi = \sqrt{\Phi^2} \quad \text{with} \quad \Phi^2 = \varphi^* \varphi, \quad (1.6)$$

the  $\Phi$ - and the  $S$ -part of  $\varphi$  played different roles in this theory with  $\Phi$  being affected by the  $D(1)$  (i.e. Weyl) gauge degree of freedom related to the  $W_4$  geometry while  $S$  being, as usual, affected by the  $U(1)$  (electromagnetic) gauge degree of freedom. Besides the fact that the energy-momentum tensor for the  $\varphi$ -field automatically appears in the modified “new improved” form (compare Callan, Coleman and Jackiw [5]) in the adopted  $W_4$  framework the modulus field  $\Phi$  could be related to the Weyl vector field  $\kappa_\rho$  and – in this way – played a geometric role in the theory. In the present paper we want to study the symmetry breaking yielding a  $V_4$  from a  $W_4$  by including an additional Dirac spinor field  $\psi$  into the theory, which is considered as a genuine matter field, and determine the mass the  $\psi$ -field acquires in the symmetry reduction involving only a mass giving term for the  $\varphi$ -field in connection with a curvature invariant of the underlying Weyl geometry (the Weyl curvature scalar,  $R$ , as already mentioned above). To study the implications of the presence of a spinor field in this Weyl geometric setting is the main theme of the present paper.

However, we do not consider in this context the case of a Weyl-geometric description of spinors alone, nor do we touch upon the question whether there exist spinor geometries of Weyl type which are not related to Riemannian geometry [6]. Since we use the Weyl-geometric framework as a mathematical tool to construct, ultimately, a physical theory with nonzero masses from a Weyl-symmetric massless theory for bosonic and fermionic fields, we are only interested in those aspects of a Weyl geometry involving spinor fields which naturally connect to a pseudo-Riemannian theory with nonzero masses for the spinor

fields. Hence the Weyl-symmetry describes a massless scenario behind the curtain of our real world which itself is characterized by a broken Weyl-symmetry with massive source terms representing ponderable matter generating gravitation as implied by Einstein's general theory of relativity. We also include electromagnetism in this description (Sect. III) and the presence of massless radiation fields before and after Weyl-symmetry breaking.

The theory we present in this paper is certainly not fully realistic. In order to make closer contact with the Higgs phenomenon of the standard model in particle physics it would be necessary to introduce an additional local weak isospin group  $SU(2)$  into the Weyl framework as well as a corresponding representation character for the scalar field and for the chiral fermion fields possessing particular hypercharges. We shall not do this in this paper in order to study the geometric mass giving phenomenon in its simplest and most transparent form for electrically charged scalar and spinor fields and determine the role played by the Weyl vector field  $\kappa_\rho$  in this context. It appears that the modulus  $\Phi$  of the scalar field  $\varphi$  acts like a Higgs field determining  $\kappa_\rho$  in the limiting  $V_4$  theory with vanishing length curvature, i.e. with  $f = 0$ . This is concluded from a consideration of the trace and divergence relations following from the Weyl-covariant equations for the metric in the unbroken theory with a Yukawa-type coupling between the  $\varphi$  and  $\psi$  fields. Moreover, an essential part in this discussion is the use of the contracted Bianchi identities for a  $W_4$  as well as the square of the Dirac operator. This investigation is carried out in Sect. II and, including electromagnetism, in Sect. III. In its broken form the divergence relations – with the physical meaning of energy and momentum balance equations – are, finally, studied in Sect. IV. The determination of the free parameters of the theory and the relation to the field equations of general relativity are given in Sect. V. We end the discussion with some final remarks on the results obtained which are presented in Sect. VI.

## II. WEYL-INVARIANT LAGRANGE THEORY

It is well-known that the scalar wave equation in a  $V_4$  for a massless field  $\varphi$  reading

$$\square\varphi + \frac{1}{6}\bar{R}\varphi = 0 \quad (2.1)$$

is conformally invariant [7], i.e. it is invariant against conformal rescalings of the metric of the underlying Riemannian space-time according to Eq. (1.2) provided the field  $\varphi$  has the conformal weight  $w(\varphi) = -\frac{1}{2}$  (in a four-dimensional space). In (2.1) the scalar curvature of the  $V_4$  is denoted by  $\bar{R}$ , and  $\square = g^{\mu\nu}\bar{\nabla}_\mu\partial_\nu$  is the d'Alembert operator. (For the notation used in this and the subsequent sections see Appendix A.)

We want to study in this paper conformal rescalings of the metric in a Weyl-geometric setting, i.e. as part of the Weyl-symmetry expressed by Eqs. (1.2) and (1.3) involving the metric  $g_{\mu\nu}$  and the Weyl vector field  $\kappa_\rho$ . We thus want to extend the field equation for a complex massless scalar field  $\varphi$  to a Weyl space  $W_4$ . At the same time we include a massless Dirac spinor field  $\psi$  as a representative of fermionic matter and investigate a Weyl-invariant dynamics of the four fields  $g_{\mu\nu}, \kappa_\rho, \varphi$  and  $\psi$ . Here the first two fields represent the geometric side of the problem while  $\varphi$  and  $\psi$  are, so to speak, bosonic and fermionic degrees of freedom representing the matter side of the dynamics. However, this division into geometric and material aspects is superficial in a massless Weyl-invariant formulation since all fields having particular nonzero Weyl weights transform nontrivially under Weyl transformations and thus the geometric fields mix with the massless  $\varphi$  and  $\psi$  fields.

The Weyl weights of  $g_{\mu\nu}$  and  $\kappa_\rho$  are 1 and 0, respectively (see Appendix A). The Weyl weight of  $\varphi$  is taken to be  $w(\varphi) = -\frac{1}{2}$ , as mentioned above, and the Weyl weight of the Dirac field  $\psi$  is chosen to be  $w(\psi) = -\frac{3}{4}$  (compare Pauli [8] in this context). This last choice is dictated by the fact that the Dirac vector current,  $j_\mu^{(\psi)}$ , (see Sect. III below) should have Weyl weight  $-1$  so that this current can act as a source current in Maxwell's equations. (The coupling of the above described dynamical system to the electromagnetic fields yielding a complete massless dynamics including electromagnetic interactions and radiation fields  $F_{\mu\nu}$ , described by a Weyl- and  $U(1)$  covariant gauge theory including gravitation, will be studied in detail in Sect. III). Ultimately, the Weyl-invariance has to be broken since there are nonzero masses and corresponding length scales appearing in nature. However,

our proposal is to approach the realistic physical world from a massless Weyl invariant description in order to investigate, from a geometric point of view, how a mass or length scale may be established in a physical theory by the breaking of a symmetry characterizing a massless situation. At the same time the capacity of matter and electromagnetic radiation of generating gravitational fields will be included in the theory as an essential structural element leading, ultimately, to Einstein's equations in the broken version of the theory.

Although our motivation in this endeavour is, as mentioned, geometric in its origin we shall use the Lagrangean method – formulated against the adopted Weyl geometric background – in order to specify a particular theory: The Lagrangean density  $\mathcal{L}_{W_4}$  should be a scalar of Weyl weight zero depending on  $g_{\mu\nu}$ ,  $\kappa_\rho$ ,  $\varphi$ , and  $\psi$  and their first derivatives to yield a set of second-order Weyl-covariant field equations characterizing the massless dynamics. We shall see that the postulated Weyl-invariance of  $\mathcal{L}_{W_4}$  and the construction of  $\mathcal{L}_{W_4}$  in terms of Weyl-covariant derivatives of the quantities involved (see Appendix A) limits the number of possibilities considerably.

The Weyl-invariant Lagrangean on which we shall base our subsequent discussion in the massless case, neglecting electromagnetism, is the following scalar hermitean density of Weyl weight zero:

$$\begin{aligned} \mathcal{L}_{W_4} = K\sqrt{-g} & \left\{ \frac{1}{2}g^{\mu\nu}D_\mu\varphi^*D_\nu\varphi - \frac{1}{12}R\varphi^*\varphi + \tilde{\alpha}R^2 - \beta(\varphi^*\varphi)^2 + \right. \\ & \left. + \frac{i}{2}(\bar{\psi}\gamma^\mu\overrightarrow{D}_\mu\psi - \bar{\psi}\overleftarrow{D}_\mu\gamma^\mu\psi) + \tilde{\gamma}\sqrt{\varphi^*\varphi}(\bar{\psi}\psi) - \tilde{\delta}\frac{1}{4}f_{\mu\nu}f^{\mu\nu} \right\}, \end{aligned} \quad (2.2)$$

with

$$D_\nu\varphi = \partial_\nu\varphi + \frac{1}{2}\kappa_\nu\varphi, \quad (2.3)$$

and

$$\overrightarrow{D}_\mu\psi = \overrightarrow{\nabla}_\mu\psi + \frac{3}{4}\kappa_\mu\psi = \left( \overrightarrow{\partial}_\mu + i\Gamma_\mu + \frac{3}{4}\kappa_\mu \right)\psi, \quad (2.4)$$

$$\bar{\psi}\overleftarrow{D}_\mu = \bar{\psi}\overleftarrow{\nabla}_\mu + \frac{3}{4}\kappa_\mu\bar{\psi} = \bar{\psi}\left( \overleftarrow{\partial}_\mu - i\Gamma_\mu + \frac{3}{4}\kappa_\mu \right) \quad (2.5)$$

being the Weyl-covariant derivatives of the scalar and spinor fields, respectively, where  $\nabla_\mu$  denotes the covariant derivative with respect to the  $W_4$ -connection with coefficients defined in (A1). Moreover

$$\Gamma_\mu = \Gamma_\mu(x) = \lambda_\mu^j(x) \frac{1}{2} \Gamma_{jik}(x) S^{ik}, \quad (2.6)$$

$$\text{with } S^{ik} = \frac{i}{4} [\gamma^i, \gamma^k] \quad (2.7)$$

is the spin connection [where the local Lorentz index  $j$  is turned into a Greek index  $\mu$  with the help of the vierbein fields  $\lambda_\mu^j(x)$ ], i.e. is the connection on the associated spinor bundle

$$S_W = S_W(W_4, \mathbf{C}_4, Spin(3, 1) \otimes D(1)), \quad (2.8)$$

with fiber  $F = \mathbf{C}_4$  being a representation space for the Dirac spinors and with the structural group  $Spin(3, 1) \otimes D(1)$ . A spinor field  $\psi(x)$  is a section of  $S_W$  with the usual local action of the group  $Spin(3, 1)$ , and with  $D(1)$  acting on the absolute value of  $\psi(x)$ , i.e. on the scalar  $\sqrt{\bar{\psi}(x)\psi(x)}$  with Weyl weight  $-\frac{3}{4}$  [compare (A2)].

The bundle  $S_W$  is associated to the spin frame bundle over  $W_4$

$$\bar{P}_W = \bar{P}_W(W_4, \bar{G} = Spin(3, 1) \otimes D(1)), \quad (2.9)$$

which is related to the Weyl frame bundle defined in Eq. (A6) in the usual way by lifting the homomorphism between the universal covering group  $Spin(3, 1)$  and the orthochronous Lorentz group  $SO(3, 1)$  to the bundles  $\bar{P}_W$  and  $P_W$ . The homomorphism between the structural groups of the two bundles (with the  $D(1)$  factor being untouched) is provided by the well-known formula

$$S(\Lambda) \gamma^i S^{-1}(\Lambda) = [\Lambda^{-1}]^i_k \gamma^k, \quad (2.10)$$

where  $S(\Lambda) \in Spin(3, 1)$ , and  $S^{-1}(\Lambda) = \gamma^0 S^\dagger(\Lambda) \gamma^0$  with  $S^\dagger(\Lambda)$  denoting the adjoint of  $S(\Lambda)$ , and  $\Lambda \in SO(3, 1)$ . The  $\Gamma_{jik}$  in Eq. (2.6) are the connection coefficients defined in Eqs. (A8) and (A10), and the  $\gamma^i; i = 0, 1, 2, 3$  are the constant Dirac matrices obeying

$$\{\gamma^i, \gamma^k\} = \gamma^i \gamma^k + \gamma^k \gamma^i = 2\eta^{ik} \mathbf{1}. \quad (2.11)$$

The  $\gamma$ -matrices with upper Greek indices appearing in (2.2) are  $x$ -dependent quantities of Weyl weight  $-\frac{1}{2}$  defined by

$$\gamma^\mu = \gamma^\mu(x) = \lambda_i^\mu(x) \gamma^i \quad \text{obeying} \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbf{1}. \quad (2.12)$$

Finally,  $R$  in (2.2) denotes the scalar curvature of the  $W_4$  having Weyl weight  $w(R) = -1$ , which is defined in (A31). All the terms in the curly brackets of Eq. (2.2) have Weyl weight  $-2$  which, together with  $w(\sqrt{-g}) = +2$ , yields Weyl weight zero for  $\mathcal{L}_{W_4}$ .

The first term in the curly brackets of (2.2) is the kinetic part for the complex  $\varphi$  field. The second term is the contribution guaranteeing conformal invariance in a  $V_4$  limit. The third term with constant  $\tilde{\alpha}$  (of dimension  $[L^2]$ ;  $L$ =length) is a term included to yield a non-trivial dynamics for the  $\kappa_\rho$ -fields. The fourth term multiplied by a constant  $\beta$  (of dimension  $[L^{-2}]$ ) is the nonlinear  $\varphi^4$ -coupling of the  $\varphi$ -field allowed by the Weyl weight  $w(\varphi) = -\frac{1}{2}$ . The fifth term is the kinetic term for the Dirac field. The sixth term is a Yukawa-like coupling of  $|\varphi| = \sqrt{\Phi^2}$  to  $(\bar{\psi}\psi)$  with constant  $\tilde{\gamma}$  (of dimension  $[L^{-1}]$ )<sup>1</sup>. Finally, the last term, multiplied by a constant  $\tilde{\delta}$  (of dimension  $[L^2]$ ), is the contribution of the  $f$ -curvature, i.e. of the free  $\kappa_\rho$  fields to the total Lagrangean. [We remark in parenthesis that the quadratic curvature invariants of Weyl weight zero for a  $W_4$  are investigated at the end of Appendix A and in Appendix C. It appears from this discussion (compare Eq. (A54)) that the invariant  $\sqrt{-g}f_{\mu\nu}f^{\mu\nu}$  plays a special role compared to the other quadratic curvature invariants for a  $W_4$ . For this reason we include in (2.2) besides the term  $\sim \sqrt{-g}R^2$  only the term  $\sim \sqrt{-g}f_{\mu\nu}f^{\mu\nu}$  in close analogy to electromagnetism.] The length dimension of the field  $\varphi$

<sup>1</sup>One could also think of writing the Yukawa-like coupling term of Weyl weight  $-2$  between the  $\varphi$ - and the  $\psi$ -fields in the form  $\tilde{\gamma}' \frac{1}{2}(\varphi + \varphi^*)(\bar{\psi}\psi)$  yielding qualitatively the same results as the  $\tilde{\gamma}$ -coupling used in the text which, however, has the advantage of being independent of the phase  $S$  of  $\varphi$ .

is here assumed to be zero. Relative to this the fermion field has length dimension  $[L^{-\frac{1}{2}}]$ . The overall constant  $K$  (with dimension [energy  $\cdot L^{-1}$ ]) is a factor converting the length dimension of the expressions in the curly brackets (which is  $[L^{-2}]$ ) into [energy  $\cdot L^{-3}$ ] to give to  $\mathcal{L}_{W_4}$  – later after symmetry breaking – the correct dimension of an energy/vol. The  $K$  factor drops out of the field equations for the Weyl-symmetric case discussed in this section and appears in (2.2) only for convenience.

We now vary the fields in Eq. (2.2) according to the usual rules and determine the field equations from the variational principle  $\delta \int \mathcal{L}_{W_4} d^4x = 0$ . Using the notation for the modulus of  $\varphi$  introduced in Eq. (1.6) the field equations in Weyl-covariant form are:

$$\delta\varphi^* : g^{\mu\nu} D_\mu D_\nu \varphi + \frac{1}{6} R \varphi + 4\beta(\varphi^* \varphi) \varphi - \tilde{\gamma} \frac{\varphi}{\sqrt{\Phi^2}} (\bar{\psi}\psi) = 0, \quad (2.13)$$

$$\delta\psi^\dagger : -i\gamma^\mu D_\mu \psi - \tilde{\gamma} \sqrt{\Phi^2} \psi = 0, \quad (2.14)$$

$$\delta\kappa_\rho : \tilde{\delta} D_\mu f^{\mu\rho} = -6\tilde{\alpha} D^\rho R, \quad (2.15)$$

$$\begin{aligned} \delta g^{\mu\nu} : & \frac{1}{6} \Phi^2 \left[ R_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu} R \right] - 4\tilde{\alpha} R \left[ R_{(\mu\nu)} - \frac{1}{4} g_{\mu\nu} R \right] - 4\tilde{\alpha} \left\{ D_{(\mu} D_{\nu)} R - g_{\mu\nu} D^\rho D_\rho R \right\} = \\ & = \Theta_{\mu\nu}^{(\varphi)} + T_{\mu\nu}^{(\psi)} + T_{\mu\nu}^{(f)} - g_{\mu\nu} \tilde{\gamma} \sqrt{\Phi^2} (\bar{\psi}\psi), \end{aligned} \quad (2.16)$$

together with the complex conjugate equation of (2.13) and the Dirac adjoint of (2.14). Here we have used in the last equation the following abbreviations for the symmetric energy-momentum tensors of Weyl weight  $-1$  for the  $\varphi$ -field, the  $\psi$ -field, and the  $f$ -field, respectively, defined in a Weyl-covariant manner:

$$\begin{aligned} \Theta_{\mu\nu}^{(\varphi)} &= \frac{1}{2} (D_\mu \varphi^* D_\nu \varphi + D_\nu \varphi^* D_\mu \varphi) - \frac{1}{6} \left\{ D_{(\mu} D_{\nu)} \Phi^2 - g_{\mu\nu} D^\rho D_\rho \Phi^2 \right\} \\ &\quad - g_{\mu\nu} \left[ \frac{1}{2} g^{\rho\lambda} D_\rho \varphi^* D_\lambda \varphi - \beta(\varphi^* \varphi)^2 \right], \end{aligned} \quad (2.17)$$

$$T_{\mu\nu}^{(\psi)} = \frac{i}{2} \left\{ \bar{\psi} \gamma_{(\mu} \overrightarrow{D}_{\nu)} \psi - \bar{\psi} \overleftarrow{D}_{(\mu} \gamma_{\nu)} \psi \right\} - g_{\mu\nu} \frac{i}{2} \left\{ \bar{\psi} \gamma^\rho \overrightarrow{D}_\rho \psi - \bar{\psi} \overleftarrow{D}_\rho \gamma^\rho \psi \right\}, \quad (2.18)$$

$$T_{\mu\nu}^{(f)} = -\tilde{\delta} \left[ f_{\mu\sigma} f_\nu^\sigma - \frac{1}{4} g_{\mu\nu} f^{\rho\lambda} f_{\rho\lambda} \right]. \quad (2.19)$$

The  $\varphi$ - and  $\psi$ -equations (2.13) and (2.14) are coupled via the  $\tilde{\gamma}$  term which lead to mass-like expressions in both equations. We shall come back below to the status of these coupling

terms in the Weyl-symmetric framework (i.e. before symmetry breaking) when we discuss the conditions following from Eq. (2.16). We remark in passing that the  $\kappa_\rho$  field actually drops out of Eqs. (2.13) and (2.14) due to the identities

$$g^{\mu\nu}D_\mu D_\nu \varphi + \frac{1}{6}R\phi = g^{\mu\nu}\bar{\nabla}_\mu \partial_\nu \varphi + \frac{1}{6}\bar{R}\varphi, \quad (2.20)$$

and

$$-i\gamma^\mu D_\mu \psi = -i\gamma^\mu \bar{\nabla}_\mu \psi, \quad (2.21)$$

which are valid for the Weyl weights chosen for  $\varphi$  and  $\psi$ , respectively. Here  $\bar{\nabla}_\mu$  denotes the metric covariant derivative [compare the discussion after Eq. (A1)]. Also on the lhs of (2.15) one can use the identity

$$D_\mu f^{\mu\rho} = \bar{\nabla}_\mu f^{\mu\rho} \quad (2.22)$$

to eliminate the  $\kappa_\rho$  contributions in the covariant derivative. Eq. (2.15) represents a Maxwell-type equation coupling the field strength  $f^{\mu\rho}$  to a current proportional to  $D^\rho R = \partial^\rho R + \kappa^\rho R$ . It is easy to show by contracting (2.15) with  $D_\rho$ , i.e. in taking the Weyl-covariant divergence, that

$$\tilde{\alpha} D^\rho D_\rho R = 0 \quad (2.23)$$

exhibiting the  $W_4$ -covariant current conservation following from (2.15) together with  $\bar{\nabla}_\rho \bar{\nabla}_\mu f^{\mu\rho} = 0$ .

The equations (2.15) and (2.16) do contain  $\kappa_\rho$  in a complicated manner. They represent 4+10 coupled Weyl-covariant equations for the determination of the Weyl vector field  $\kappa_\rho$  and the metric  $g_{\mu\nu}$  [modulo Weyl-transformations (1.2), (1.3)]. The most complicated field equations are the ten equations (2.16) which are of Einstein type relating  $W_4$  curvature expressions (on the lhs) to the energy-momentum tensors of the fields involved in the dynamics (on the rhs) and an additional coupling term proportional to  $\tilde{\gamma}$ . However, no gravitational constant is identifiable in these scaleless Weyl-covariant equations.

Moreover, regarding the expression for  $\Theta_{\mu\nu}^{(\varphi)}$  we like to point out that the energy-momentum tensor for the  $\varphi$  field here automatically appears in the “new improved” form including the terms in the curly brackets (compare Ref. [5]). In our context these additional contributions expressed in terms of  $\Phi^2$ , with Weyl weight  $-1$ , appear of course in Weyl-covariant form, i.e. constructed with Weyl-covariant derivatives. This “new improved” addition to the conventional energy momentum tensor,  $T_{\mu\nu}^{(\varphi)}$ , for the  $\varphi$  field, which was introduced by Callan, Coleman and Jackiw at first in a  $\varphi^4$ -theory formulated in flat Minkowski space, originates here from the variation of the second term in the Lagrangean (2.2) involving the  $W_4$  curvature scalar, i.e. from the term in the variational integral

$$-K\sqrt{-g}\frac{1}{12}\Phi^2\delta R = -K\sqrt{-g}\frac{1}{2}\Phi^2\left[(\delta\bar{R}_{\mu\nu} + \delta P_{\mu\nu})g^{\mu\nu} + (\bar{R}_{\mu\nu} + P_{\mu\nu})\delta g^{\mu\nu}\right]. \quad (2.24)$$

Let us next check the trace condition following from (2.16). In order to find the trace of (2.16) we first compute the traces of the energy momentum tensors (2.17) – (2.19) for solutions of the field equations, i.e. by using the equations (2.13) – (2.15) in the derivation. The result is

$$\Theta_{\mu}^{(\varphi)\mu} = -\frac{1}{6}\Phi^2R + \tilde{\gamma}\sqrt{\Phi^2}(\bar{\psi}\psi), \quad (2.25)$$

$$T_{\mu}^{(\psi)\mu} = 3\tilde{\gamma}\sqrt{\Phi^2}(\bar{\psi}\psi), \quad (2.26)$$

$$T_{\mu}^{(f)\mu} = 0. \quad (2.27)$$

For the trace of (2.16) one now finds with the help of (2.23) and (2.25) – (2.27) that the  $\tilde{\gamma}$  term drops out of this equation and that the rest is identically satisfied yielding thus no further constraints. This situation may change when we consider the Weyl-covariant divergence of (2.16) and make use of the contracted Bianchi identities for a  $W_4$  for the geometric quantities. Then four energy and momentum balance equations are obtained which could contain additional information worth to be extracted.

In order to compute the Weyl-covariant divergence of (2.16) we need the corresponding divergences of Eqs.(2.17) – (2.19), again for solutions of the field equations (2.13) –

(2.15). Using these as well as the formulae (A33) - (A36) of Appendix A one finds by a straightforward but lengthly calculation the following results:

$$D^\mu \Theta_{\mu\nu}^{(\varphi)} = \frac{1}{6} \left[ R_{(\nu\rho)} - \frac{1}{2} g_{\nu\rho} R \right] D^\rho \Phi^2 - \frac{1}{12} \Phi^2 D^\rho f_{\rho\nu} + \tilde{\gamma} (D_\nu \sqrt{\Phi^2}) (\bar{\psi} \psi), \quad (2.28)$$

$$\begin{aligned} D^\mu T_{\mu\nu}^{(\psi)} &= -\frac{1}{4} \left( \bar{\psi} [\gamma^\mu \mathcal{R}_{\mu\nu} + \mathcal{R}_{\mu\nu} \gamma^\mu] \psi \right) - \frac{1}{16} \left( \bar{\psi} [\gamma_\nu S^{\mu\rho} + S^{\mu\rho} \gamma_\nu] \psi \right) f_{\mu\rho} \\ &\quad + \tilde{\gamma} \sqrt{\Phi^2} D_\nu (\bar{\psi} \psi), \end{aligned} \quad (2.29)$$

$$D^\mu T_{\mu\nu}^{(f)} = 6\tilde{\alpha} f_{\nu\rho} D^\rho R, \quad (2.30)$$

with  $\mathcal{R}_{\mu\nu}$  as defined in Eq. (2.39) below. One can show that the first two terms in (2.29) each have the form of the coupling of a dual curvature,  ${}^*f_{\mu\nu}$ , to an axial vector current  $(\bar{\psi} \gamma^\nu \gamma^5 \psi)$  of the spinor field representing the work done by the dual  $f$ -curvature field on the axial vector current of the  $\psi$ -field given by the expression  ${}^*f_{\mu\nu} (\bar{\psi} \gamma^\nu \gamma^5 \psi)$  times a numerical constant. In fact, in order to rewrite Eq. (2.29) in a form involving the hermitean axial vector current  $(\bar{\psi} \gamma^\nu \gamma^5 \psi)$  of the  $\psi$  field one has to use Eq. (2.40) below and the formula

$$\{\gamma^k, S^{ij}\} = \gamma^k S^{ij} + S^{ij} \gamma^k = \varepsilon^{kij}_l \gamma^l \gamma^5, \quad (2.31)$$

where  $\varepsilon^{kijl}$  is the Levi-Civita tensor with  $\varepsilon_{0123} = +1$ , and

$$\gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \quad \text{with} \quad \gamma^{5\dagger} = \gamma^5. \quad (2.32)$$

The dual  $f$ -curvature tensor is defined by

$${}^*f_{ik} = \frac{1}{2} \varepsilon_{ikjl} f^{jl} \quad (2.33)$$

with the corresponding Greek indexed quantities being, as usual, obtained by conversion with the help of the vierbein fields yielding  ${}^*f_{\nu\rho} = \frac{1}{2} \sqrt{-g} \varepsilon_{\nu\rho\mu\sigma} f^{\mu\sigma}$  possessing Weyl weight zero. Here the following transformation rule for the  $\varepsilon$ -tensor is to be used:

$$\varepsilon_{ijkl} = \lambda_i^\mu \lambda_j^\nu \lambda_k^\rho \lambda_l^\sigma \sqrt{-g} \varepsilon_{\mu\nu\rho\sigma}. \quad (2.34)$$

After these changes in (2.29) one now obtains with the help of Eqs. (2.28) – (2.30) and the twice contracted Bianchi identities in the form of Eq. (A40) from Eqs. (2.16) for

solutions of the field equations (2.13) – (2.15) a set of covariant divergence relations which are *identically satisfied*. Thus no constraints arise in the Weyl-symmetric theory defined by the Lagrangean (2.2) from the work done by the  ${}^*f$ -curvature field on the axial vector current of the spinor field, i.e. the first two terms in (2.29), in fact, cancel. Also the contributions of the scalar field disappear from the divergence relations, and the terms proportional to  $\tilde{\gamma}$  resulting from the Yukawa-like coupling between the  $\varphi$ - and the  $\psi$ -field cancel as well in these relations. Hence the energy and momentum balance in our Weyl-symmetric massless theory is automatically satisfied for nonvanishing  $f$ - and  ${}^*f$ -curvature. This situation changes when the Weyl-symmetry is broken in Sect. IV.

We, finally, mention that for the Weyl-covariant divergence of the axial vector current  $(\bar{\psi}\gamma_\mu\gamma^5\psi)$  of Weyl weight  $-1$  one obtains from the Dirac equation (2.14) and its adjoint the result:

$$D^\mu(\bar{\psi}\gamma_\mu\gamma^5\psi) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\bar{\psi}\gamma^\mu\gamma^5\psi) = -2i\tilde{\gamma}\sqrt{\Phi^2}(\bar{\psi}\gamma^5\psi), \quad (2.35)$$

where the first equality is due to the fact that the  $\kappa_\rho$ -contribution of the connection coefficients (A1) cancels against the Weyl weight contribution for a quantity of Weyl weight  $-1$  [compare Eq. (B6)]. Eq. (2.35) shows that the axial vector current of the Dirac spinor field has nonvanishing divergence provided  $\tilde{\gamma} \neq 0$ .

We conclude this section by considering the square of the Dirac operator

$$\not{D} = -i\gamma^\mu D_\mu, \quad (2.36)$$

where  $\not{D}$  is a matrix-valued operator of Weyl weight  $-\frac{1}{2}$ . Let us consider the square of  $\not{D}$  applied to a Dirac field  $\psi$  of Weyl weight  $w(\psi) = -\frac{3}{4}$  [compare Schrödinger [9] for the Riemannian case]:

$$\not{D}\not{D}\psi = -g^{\mu\nu}D_\mu D_\nu\psi - \frac{1}{4}R\psi + \frac{i}{4}f_{\mu\nu}S^{\mu\nu}\psi. \quad (2.37)$$

To derive Eq. (2.37) one uses the cyclic identities (A36) together with the equation

$$[D_\mu, D_\nu]\psi = i\mathcal{R}_{\mu\nu}\psi - w(\psi)f_{\mu\nu}\psi, \quad (2.38)$$

where

$$\mathcal{R}_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + i[\Gamma_\mu, \Gamma_\nu] \quad (2.39)$$

is the matrix-valued  $W_4$ -curvature associated with the spin connection  $\Gamma_\nu$ , defined in (2.6), which is related to the curvature tensor defined in (A19) by

$$\mathcal{R}_{\mu\nu} = \lambda_\mu^k \lambda_\nu^l \frac{1}{2} R_{kl ij} S^{ij}. \quad (2.40)$$

To obtain (2.37) one, moreover, needs the relation

$$D_\rho \gamma^\mu = \nabla_\rho \gamma^\mu + \frac{1}{2} \kappa_\rho \gamma^\mu + i[\Gamma_\rho, \gamma^\mu] = 0 \quad (2.41)$$

which is analogous to the equation (A9) for the vierbein fields  $\lambda_i^\mu$  yielding, with (2.12),  $D_\rho g^{\mu\nu} = 0$  [compare (A4)]. The derivation of (2.37) is facilitated by making use of the Bach tensor [compare Eqs. (A41) – (A43)] possessing the same symmetries as the Riemann-Christoffel tensor  $\bar{R}_{\mu\nu\rho\sigma}$  in a  $V_4$ .

Using now the Dirac equation (2.14) we compute for the lhs of (2.37)

$$\not D \not D \psi = \tilde{\gamma} \left\{ -i \gamma^\mu \psi D_\mu \sqrt{\Phi^2} + \tilde{\gamma} \Phi^2 \psi \right\}. \quad (2.42)$$

If we now demand that the squared Dirac operator has a sharp nonzero eigenvalue originating from the Yukawa-like coupling proportional to  $\tilde{\gamma}$ , and that this eigenvalue is the same for all components of  $\psi$ , we have to conclude that the first term on the rhs of (2.42), being nondiagonal in spin space, vanishes requiring that  $\Phi^2$  is covariant constant, i.e.

$$D_\mu \Phi^2 = \partial_\mu \Phi^2 + \kappa_\mu \Phi^2 = 0. \quad (2.43)$$

In this case (2.42) together with (2.37) would read, suppressing the  $f_{\mu\nu}$ -term which is zero in this case (see below),

$$\not D \not D \psi = -g^{\mu\nu} D_\mu D_\nu \psi - \frac{1}{4} R \psi = \tilde{\gamma}^2 \Phi^2 \psi \quad (2.44)$$

where, for constant modulus  $\Phi$  (see Sect. IV and V below),  $\tilde{\gamma}^2 \Phi^2$  would play the role of a quadratic mass term,  $\left(\frac{Mc}{\hbar}\right)^2$ , for the  $\psi$  field. However,  $D_\mu \Phi^2 = 0$  – which is a typical symmetry breaking relation – implies according to (2.43)

$$\kappa_\rho = -\partial_\rho \log \Phi^2, \quad (2.45)$$

expressing the fact that the  $\kappa_\rho$ -field is “pure gauge” and, hence, the  $f$ -curvature is vanishing. For this reason we suppressed above the  $f_{\mu\nu}$ -term in going from (2.37) to (2.44). Therefore only in the limiting case of a  $W_4$  reducing to a  $V_4$  can the square of the Dirac operator be a diagonal operator with eigenvalue  $\tilde{\gamma}^2 \Phi^2$ . We shall come back to this property in Sect. IV below when we consider the explicit breaking of the Weyl-symmetry.

### III. MASSLESS WEYL-INVARIANT DESCRIPTION INCLUDING ELECTROMAGNETISM

In this section we extend the theory presented in Sec. II and include the electromagnetic interaction yielding a  $D(1)$  and  $U(1)$  gauge invariant massless theory containing also a dynamics for the metric in the form of ten Weyl-covariant equations for the metric tensor  $g_{\mu\nu}$  similar to the ones obtained in the previous section. These field equations (see Eqs. (3.12) below) are of general relativistic type satisfying the general covariance principle besides their  $D(1)$  and  $U(1)$  gauge covariance. However, as already mentioned above, they contain no gravitational coupling constant. Such a constant can only be identified after the Weyl-symmetry has been broken to yield a Riemannian description in the limit (see Sects. IV and V below). This is so because Einstein’s field equations in general relativity coupled to material sources are not conformally invariant.

To include electromagnetism in a Weyl-invariant scheme the respective fiber bundles introduced so far are generalized by adding an additional  $U(1)$  fiber and extending the structural groups of  $P_W$  and  $\bar{P}_W$  to  $SO(3, 1) \otimes D(1) \otimes U(1)$  and  $Spin(3, 1) \otimes D(1) \otimes U(1)$ , respectively. We call the resulting principal bundles  $\tilde{P}_W$  and  $\tilde{\bar{P}}_W$ , respectively, with a pull back of a connection on  $\tilde{P}_W$  being given by the set of one-forms ( $\omega_{ik} = -\omega_{ki}, \kappa, A$ ) and the corresponding curvature two-forms ( $\Omega_{ik} = -\Omega_{ki}, f, F$ ), and analogously for  $\tilde{\bar{P}}_W$ . In a natural basis one has, in addition to the relations of Appendix A:

$$A = A_\mu dx^\mu, \quad F = dA = \frac{1}{2} dx^\mu \wedge dx^\nu F_{\mu\nu}, \quad (3.1)$$

with the electromagnetic potentials  $A_\mu$  of Weyl weight zero and the electromagnetic field strengths  $F_{\mu\nu}$  [ $U(1)$  or  $F$ -curvature] of Weyl weight zero, the latter being given by

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (3.2)$$

obeying

$$D_{\{\lambda} F_{\mu\nu\}} = \partial_{\{\lambda} F_{\mu\nu\}} = 0. \quad (3.3)$$

In (3.2) and (3.3) the Weyl-covariant derivatives could be replaced by the ordinary ones due to the symmetry of the connection coefficients defined in (A1).

The new Lagrangean, called  $\tilde{\mathcal{L}}_{W_4}$ , is obtained from  $\mathcal{L}_{W_4}$ , defined in (2.2), by the minimal substitution of the Weyl-covariant derivatives,  $D_\nu$ , by  $D(1)$  and  $U(1)$  as well as generally covariant derivatives,  $\tilde{D}_\mu$ , i.e.

$$D_\mu \varphi \longrightarrow \tilde{D}_\mu \varphi = D_\mu \varphi + \frac{iq}{\hbar c} A_\mu \varphi, \quad (3.4)$$

$$D_\mu \varphi^* \longrightarrow \tilde{D}_\mu \varphi^* = D_\mu \varphi^* - \frac{iq}{\hbar c} A_\mu \varphi^*, \quad (3.5)$$

and analogously

$$\overrightarrow{D}_\mu \psi \longrightarrow \overrightarrow{\tilde{D}}_\mu \psi = \left( \overrightarrow{D}_\mu + \frac{ie}{\hbar c} A_\mu \cdot \mathbf{1} \right) \psi, \quad (3.6)$$

$$\bar{\psi} \overleftarrow{D}_\mu \longrightarrow \bar{\psi} \overleftarrow{\tilde{D}}_\mu = \bar{\psi} \left( \overleftarrow{D}_\mu - \frac{ie}{\hbar c} A_\mu \cdot \mathbf{1} \right). \quad (3.7)$$

Here we have denoted the charge of the field  $\varphi$  by  $q$  and that of  $\varphi^*$  by  $-q$ , while the charge of the fermion field  $\psi$  is denoted by  $e$  and that of its adjoint by  $-e$ . In addition to the substitutions (3.4) – (3.7) we have to add, in the familiar way, a contribution for the free electromagnetic fields  $F_{\mu\nu}$  in the Lagrangean  $\tilde{\mathcal{L}}_{W_4}$ .

We remark in passing that in the minimal electromagnetic interaction, by introducing electromagnetism through the substitutions (3.4) – (3.7) into a supposedly known system of fields in quantum field theory, one usually regards the potentials  $A_\mu$  and the corresponding

fields  $F_{\mu\nu}$  as *external fields*. In the present case, however, there are no external fields the sources of which are not included in the system. We want to consider a *closed system* in which the electromagnetic fields are generated by currents of the  $\varphi$  and  $\psi$  fields, respectively. The potentials  $A_\mu$  thus describe besides the electromagnetic action of the field  $\varphi$  onto  $\psi$  and vice versa also the back reaction onto their own motion. We shall not investigate these back reactions in detail here. For a thorough discussion of the self-field contributions in the relativistic theory of classical charged point particles we refer to the book by Rohrlich [10].

Having said that the introduction of electromagnetic interactions should lead to a closed system of fields described, moreover, in the proposed Weyl-covariant manner, we hasten to add that it is certainly not without problems to introduce electromagnetic fields into a massless theory where the  $A_\mu$  fields are coupled to source currents of *massless*  $\varphi$  and  $\psi$  fields. It is very likely that a nonsingular description is only possible for  $q = e = 0$  in a massless theory. Our excuse for nevertheless introducing electromagnetism in the standard manner into our Weyl-invariant theory developed in Sect. II is that we, ultimately, intend to break the Weyl symmetry explicitly in order to generate masses for the  $\varphi$  and  $\psi$  fields. In this scenario the extension of a broken Weyl theory without electromagnetism to a broken Weyl theory including electromagnetism is correctly given by the standard arguments exhibited by the substitutions (3.4) – (3.7). After these remarks we base our further discussion in this section on the following Weyl- and  $U(1)$ -invariant Lagrangean density:

$$\begin{aligned} \tilde{\mathcal{L}}_{W_4} = K\sqrt{-g} & \left\{ \frac{1}{2}g^{\mu\nu}\tilde{D}_\mu\varphi^*\tilde{D}_\nu\varphi - \frac{1}{12}R\varphi^*\varphi - \beta(\varphi^*\varphi)^2 + \tilde{\alpha}R^2 + \frac{i}{2}(\bar{\psi}\gamma^\mu\vec{D}_\mu\psi - \bar{\psi}\overleftrightarrow{\tilde{D}}_\mu\gamma^\mu\psi) \right. \\ & \left. + \tilde{\gamma}\sqrt{\varphi^*\varphi}(\bar{\psi}\psi) - \tilde{\delta}\frac{1}{4}f_{\mu\nu}f^{\mu\nu} - \frac{1}{K}\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \right\}. \end{aligned} \quad (3.8)$$

In the usually adopted units in which  $\frac{e}{\hbar c}A_\mu$  has the dimension of an inverse length the term  $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$  has the dimension of an energy/volume. In order to convert this into a quantity of dimension  $L^{-2}$ , like all the other terms in the curly brackets of (3.8), one has to multiply the last term by a factor  $K^{-1}$  (compare in this context the discussion in Sect. II above).

The Weyl- and  $U(1)$ -covariant field equations following from a variational principle based

on the Lagrangean (3.8) are now given by:

$$\delta\varphi^* : g^{\mu\nu}\tilde{D}_\mu\tilde{D}_\nu\varphi + \frac{1}{6}R\varphi + 4\beta(\varphi^*\varphi)\varphi - \tilde{\gamma}\frac{\varphi}{\sqrt{\Phi^2}}(\bar{\psi}\psi) = 0, \quad (3.9)$$

$$\delta\psi^\dagger : -i\gamma^\mu\tilde{D}_\mu\psi - \tilde{\gamma}\sqrt{\Phi^2}\psi = 0, \quad (3.10)$$

$$\delta\kappa_\rho : \tilde{\delta}D_\mu f^{\mu\rho} = -6\tilde{\alpha}D^\rho R, \quad (3.11)$$

$$\begin{aligned} \delta g^{\mu\nu} : \frac{1}{6}\Phi^2 & \left[ R_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}R \right] - 4\tilde{\alpha}R & \left[ R_{(\mu\nu)} - \frac{1}{4}g_{\mu\nu}R \right] - 4\tilde{\alpha}\left\{ D_{(\mu}D_{\nu)}R - g_{\mu\nu}D^\rho D_\rho R \right\} = \\ & = \tilde{\Theta}_{\mu\nu}^{(\varphi)} + \tilde{T}_{\mu\nu}^{(\psi)} + T_{\mu\nu}^{(f)} + T_{\mu\nu}^{(F)} - g_{\mu\nu}\tilde{\gamma}\sqrt{\Phi^2}(\bar{\psi}\psi), \end{aligned} \quad (3.12)$$

$$\delta A_\rho : D_\mu F^{\mu\rho} = \frac{K}{\hbar c} \left[ qj^{(\varphi)\rho} + ej^{(\psi)\rho} \right]. \quad (3.13)$$

Here the  $D(1)$  and  $U(1)$  gauge covariant hermitean currents are defined by (pulling the index  $\rho$  down with  $g_{\rho\mu}$ ):

$$j_\mu^{(\varphi)} = \frac{i}{2}[\varphi^* \cdot \tilde{D}_\mu\varphi - \tilde{D}_\mu\varphi^* \cdot \varphi], \quad (3.14)$$

$$j_\mu^{(\psi)} = (\bar{\psi}\gamma_\mu\psi). \quad (3.15)$$

Both currents possess Weyl weight  $-1$ . Remember that the Weyl weight of  $\psi$  was chosen to be  $w(j_\mu^{(\psi)}) = -1$  so that the Dirac vector current (3.15) can act as a source current in Maxwell's equations (3.13), where again we can replace the lhs by [compare (2.22)]

$$D_\mu F^{\mu\rho} = \bar{\nabla}_\mu F^{\mu\rho} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g} F^{\mu\rho}) \quad (3.16)$$

being thus a quantity of Weyl weight  $-1$  independent of  $\kappa_\rho$ . The  $\kappa_\rho$ -contributions also disappear from the rhs of (3.14) with only the electromagnetic contributions remaining as in the usual Klein-Gordon theory with electromagnetic interaction. Eq. (3.11) and the definition of  $T_{\mu\nu}^{(f)}$  are unchanged [see Eqs. (2.15) and (2.19)], while  $\tilde{\Theta}_{\mu\nu}^{(\varphi)}$ ,  $\tilde{T}_{\mu\nu}^{(\psi)}$  and  $T_{\mu\nu}^{(F)}$  are given by the following  $D(1)$  and  $U(1)$  gauge covariant expressions symmetric in  $\mu$  and  $\nu$ :

$$\begin{aligned} \tilde{\Theta}_{\mu\nu}^{(\varphi)} = & \frac{1}{2}(\tilde{D}_\mu\varphi^*\tilde{D}_\nu\varphi + \tilde{D}_\nu\varphi^*\tilde{D}_\mu\varphi) - \frac{1}{6}\left\{ D_{(\mu}D_{\nu)}\Phi^2 - g_{\mu\nu}D^\rho D_\rho\Phi^2 \right\} \\ & - g_{\mu\nu}\left[ \frac{1}{2}g^{\rho\lambda}\tilde{D}_\rho\varphi^*\tilde{D}_\lambda\varphi - \beta(\varphi^*\varphi)^2 \right], \end{aligned} \quad (3.17)$$

$$\tilde{T}_{\mu\nu}^{(\psi)} = \frac{i}{2}\left\{ \bar{\psi}\gamma_{(\mu}\overrightarrow{\tilde{D}}_{\nu)}\psi - \bar{\psi}\overleftarrow{\tilde{D}}_{(\mu}\gamma_{\nu)}\psi \right\} - g_{\mu\nu}\frac{i}{2}\left\{ \bar{\psi}\gamma^\rho\overrightarrow{\tilde{D}}_\rho\psi - \bar{\psi}\overleftarrow{\tilde{D}}_\rho\gamma^\rho\psi \right\}, \quad (3.18)$$

$$T_{\mu\nu}^{(F)} = -\frac{1}{K}\left[ F_{\mu\sigma}F_\nu^{\sigma} - \frac{1}{4}g_{\mu\nu}F^{\rho\lambda}F_{\rho\lambda} \right]. \quad (3.19)$$

The total electromagnetic current appearing as source current on the rhs of (3.13) is conserved as a consequence of (3.13) and the relation  $D_\rho D_\mu F^{\mu\rho} = 0$ . On the other hand, using (3.9) and its complex conjugate, one concludes from (3.14) that the  $\varphi$ -current alone is conserved. Similarly, one concludes from (3.10) and its adjoint that the vector current (3.15) is conserved, i.e. we have the two separate charge conservation equations

$$D^\mu j_\mu^{(\varphi)} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\rho} j_\rho^{(\varphi)}) = 0, \quad (3.20)$$

$$D^\mu j_\mu^{(\psi)} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\rho} j_\rho^{(\psi)}) = 0, \quad (3.21)$$

where the  $\kappa_\rho$ -contributions in the connection coefficients (A1) and the Weyl weight term cancel for a vector of Weyl weight  $-1$  as mentioned before.

The axial vector current of the Dirac field again satisfies the divergence relation (2.35) for the theory including electromagnetism. Moreover, since Eq. (3.11) is the same as (2.15), also the relation (2.23) remains valid in the theory based on the Lagrangean  $\tilde{\mathcal{L}}_{W_4}$ . Clearly, there is no coupling between the two gauge fields  $\kappa_\rho$  and  $A_\rho$  for a gauge theory which is of direct product type.

The discussion of the trace of Eqs. (3.12) leads to the same result as in Sect. II since the additionally appearing trace of the electromagnetic energy-momentum tensor (3.19) is zero:

$$T_\mu^{(F)\mu} = 0. \quad (3.22)$$

We, finally, discuss the divergence relations following from Eqs. (3.12) including electromagnetic interactions. Using the following formula for the commutator of the covariant derivatives  $\tilde{D}_\mu$ , analogous to Eqs. (A34) and (A35) but including electromagnetism:

$$[\tilde{D}_\mu, \tilde{D}_\nu] = [D_\mu, D_\nu] + \frac{ie}{\hbar c} F_{\mu\nu}, \quad (3.23)$$

one now finds instead of Eqs. (2.28) and (2.29) – the latter after computing the again vanishing axial vector coupling resulting from the first two terms on the rhs – for the theory based on  $\tilde{\mathcal{L}}_{W_4}$ , and for the solutions of the field equations (3.9) – (3.13), the following relations:

$$D^\mu \tilde{\Theta}_{\mu\nu}^{(\varphi)} = \frac{1}{6} \left[ R_{(\nu\rho)} - \frac{1}{2} g_{\nu\rho} R \right] D^\rho \Phi^2 - \frac{1}{12} \Phi^2 D^\rho f_{\rho\nu} + \tilde{\gamma} (D_\nu \sqrt{\Phi^2}) (\bar{\psi} \psi) + \frac{q}{\hbar c} F_{\nu\rho} j^{(\varphi)\rho}, \quad (3.24)$$

$$D^\mu \tilde{T}_{\mu\nu}^{(\psi)} = \tilde{\gamma} \sqrt{\Phi^2} D_\nu (\bar{\psi} \psi) + \frac{e}{\hbar c} F_{\nu\rho} j^{(\psi)\rho}, \quad (3.25)$$

with (2.30) remaining unchanged, i.e.

$$D^\mu T_{\mu\nu}^{(f)} = 6\tilde{\alpha} f_{\nu\rho} D^\rho R, \quad (3.26)$$

and

$$D^\mu T_{\mu\nu}^{(F)} = -F_\nu^\rho \left[ \frac{q}{\hbar c} j_\rho^{(\varphi)} + \frac{e}{\hbar c} j_\rho^{(\psi)} \right]. \quad (3.27)$$

Here and in Eqs. (3.24) and (3.25) the currents  $j_\rho^{(\varphi)}$  and  $j_\rho^{(\psi)}$  are defined by Eqs. (3.14) and (3.15), respectively, while  $\tilde{\Theta}_{\mu\nu}^{(\varphi)}, \tilde{T}_{\mu\nu}^{(\psi)}$  and  $T_{\mu\nu}^{(F)}$  are defined in Eqs.(3.17), (3.18) and (3.19), respectively. With the help of Eqs. (3.24) – (3.27) the divergence relations following from (3.12), including electromagnetic effects, are now again identically satisfied showing that also the electromagnetic contributions appearing on the rhs of Eqs. (3.24), (3.25), and (3.27) cancel in these energy and momentum balance relations implying that one is, indeed, considering an electromagnetically closed system of fields.

Computing finally again the square of the Dirac operator  $\tilde{\mathcal{D}} = -i\gamma^\mu \tilde{D}_\mu$  one obtains using (3.23) together with (2.38) along the same lines as in Sect. II the  $U(1)$  and Weyl-covariant result for  $w(\psi) = -\frac{3}{4}$ :

$$\tilde{\mathcal{D}} \tilde{\mathcal{D}} \psi = -g^{\mu\nu} \tilde{D}_\mu \tilde{D}_\nu \psi - \frac{1}{4} R \psi + \frac{i}{4} f_{\mu\nu} S^{\mu\nu} \psi - \frac{e}{\hbar c} F_{\mu\nu} S^{\mu\nu} \psi. \quad (3.28)$$

#### IV. WEYL-SYMMETRY BREAKING

Having formulated the theory of a massless scalar field and a massless fermion field in a Weyl space  $W_4$  in the presence of electromagnetic fields, we now break the Weyl-symmetry by a term in the Lagrangean constructed with the help of the curvature scalar  $R$  of a  $W_4$  and a mass term for the  $\varphi$  field regarding the modulus of the scalar field as a Higgs-type field. Due to the Yukawa-like coupling proportional to  $\tilde{\gamma}$  of the  $\varphi$ - and the  $\psi$ -fields there

appeared already in the Weyl-symmetric theory treated in Sects. II and III a mass-like term for the fermion field in the field equations for  $\psi$  [see Eqs. (2.14) and (3.10)] which, in the broken theory, when the  $W_4$  reduces to a  $V_4$ , will yield a mass term proportional to  $\tilde{\gamma}$  for the  $\psi$  field. The total Lagrangean density on which we shall base the discussion in this section is thus given by

$$\mathcal{L} = \tilde{\mathcal{L}}_{W_4} + \mathcal{L}_B , \quad (4.1)$$

where  $\tilde{\mathcal{L}}_{W_4}$  is the Weyl-invariant Lagrangean defined in (3.8) and  $\mathcal{L}_B$  is the  $U(1)$  gauge invariant but Weyl-symmetry breaking term of Weyl weight +1 defined by

$$\mathcal{L}_B = -\frac{a}{2}K\sqrt{-g}\left\{\frac{1}{6}R + \left[\frac{mc}{\hbar}\right]^2\varphi^*\varphi\right\} . \quad (4.2)$$

The constant  $a$  has the length dimension [ $L^0$ ]. The symmetry breaking Lagrangean (4.2) introduces a length scale into the theory given by the Compton wave length associated with the mass of the  $\varphi$  field, and this scale breaking is related to the geometry of the underlying Weyl space being determined by the scalar curvature  $R$  of the  $W_4$  defined in (A31). The Weyl geometry thus provides the geometric framework for the breaking mechanism introducing nonzero masses for the scalar field which itself is not to be regarded as a true, bona fide, matter field. We shall see that the modulus  $\Phi$  of  $\varphi$  behaves like a geometric quantity in this formalism with  $\Phi^2$  acting as a potential for the Weyl vector fields  $\kappa_\rho$ . The role of the geometry in the mass giving procedure as described here is that of an embedding stratum or medium which takes part in the phenomenon. We thus refer to this form of mass generation for short as the “Archimedes principle”, since the ambient geometry is taking an active part in it.

Let us now write down the field equations following from a variational principle formulated with the Lagrangean  $\mathcal{L}$ . One finds, using the same notation as before:

$$\delta\varphi^* : g^{\mu\nu}\tilde{D}_\mu\tilde{D}_\nu\varphi + \frac{1}{6}R\varphi + 4\beta(\varphi^*\varphi)\varphi - \tilde{\gamma}\frac{\varphi}{\sqrt{\Phi^2}}(\bar{\psi}\psi) + a\left[\frac{mc}{\hbar}\right]^2\varphi = 0 , \quad (4.3)$$

$$\delta\psi^\dagger : -i\gamma^\mu\tilde{D}_\mu\psi - \tilde{\gamma}\sqrt{\Phi^2}\psi = 0 , \quad (4.4)$$

$$\delta\kappa_\rho : \tilde{\delta}D_\mu f^{\mu\rho} = -6\tilde{\alpha}D^\rho R + \frac{a}{4}\kappa^\rho , \quad (4.5)$$

$$\begin{aligned} \delta g^{\mu\nu} : & \frac{1}{6}(\Phi^2 + a) \left[ R_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}R \right] - 4\tilde{\alpha}R \left[ R_{(\mu\nu)} - \frac{1}{4}g_{\mu\nu}R \right] - 4\tilde{\alpha} \left\{ D_{(\mu}D_{\nu)}R - g_{\mu\nu}D^\rho D_\rho R \right\} = \\ & = \tilde{\Theta}_{\mu\nu}^{(\varphi)} + \tilde{T}_{\mu\nu}^{(\psi)} + T_{\mu\nu}^{(F)} + T_{\mu\nu}^{(F)} - g_{\mu\nu}\tilde{\gamma}\sqrt{\Phi^2}(\bar{\psi}\psi) + g_{\mu\nu}\frac{a}{2}\left[\frac{mc}{\hbar}\right]^2\Phi^2 , \end{aligned} \quad (4.6)$$

$$\delta A_\rho : D_\mu F^{\mu\rho} = \frac{K}{\hbar c} \left[ qj^{(\varphi)\rho} + ej^{(\psi)\rho} \right] , \quad (4.7)$$

$$\delta a : \frac{1}{6}R + \left[\frac{mc}{\hbar}\right]^2\Phi^2 = 0 . \quad (4.8)$$

Computing now first the traces of  $\tilde{\Theta}_{\mu\nu}^{(\varphi)}$  and  $\tilde{T}_{\mu\nu}^{(\psi)}$  using the field equations (4.3) and (4.4), respectively, one finds

$$\tilde{\Theta}_\mu^{(\varphi)\mu} = -\frac{1}{6}\Phi^2R + \tilde{\gamma}\sqrt{\Phi^2}(\bar{\psi}\psi) - a\left[\frac{mc}{\hbar}\right]^2\Phi^2 , \quad (4.9)$$

and

$$\tilde{T}_\mu^{(\psi)\mu} = 3\tilde{\gamma}\sqrt{\Phi^2}(\bar{\psi}\psi) . \quad (4.10)$$

With these results one obtains from (4.6) by contracting this equation with  $g^{\mu\nu}$  and using the field equations as well as the constraint (4.8) and Eqs. (2.27) and (3.22) that

$$\tilde{\alpha}D^\rho D_\rho R = 0 , \quad (4.11)$$

which is an equation we had obtained before in Sects. II and III from the  $\kappa_\rho$ -equations [compare Eq. (2.23)]. Now we conclude from (4.5) by taking the Weyl-covariant divergence and using (4.11) that, for  $a \neq 0$ , the Weyl vector fields must satisfy the Lorentz-type condition

$$D^\rho\kappa_\rho = 0 . \quad (4.12)$$

This equation may, with the help of (B6), be turned into the following  $V_4$  covariant form:

$$\bar{\nabla}^\rho\kappa_\rho = \frac{1}{\sqrt{-g}}\partial_\rho(\sqrt{-g}\kappa^\rho) = \kappa_\rho\kappa^\rho . \quad (4.13)$$

Eq. (4.13) implies that the  $W_4$  curvature scalar may now be expressed as

$$R = \bar{R} - \frac{3}{2}\kappa_\rho\kappa^\rho , \quad (4.14)$$

and  $R_{(\mu\nu)}$  is given by

$$R_{(\mu\nu)} = \bar{R}_{\mu\nu} + P_{(\mu\nu)} = \bar{R}_{\mu\nu} - \frac{1}{2}(\bar{\nabla}_\mu\kappa_\nu + \bar{\nabla}_\nu\kappa_\mu) - \frac{1}{2}\kappa_\mu\kappa_\nu . \quad (4.15)$$

Let us now turn to the divergence relations following from (4.6) for the solutions of the field equations. We know from our previous discussions in Sect. II and III that these energy and momentum balance relations, obtained after using the contracted Bianchi identities, were satisfied identically in the unbroken theory. Collecting now the  $a$ -dependent terms in the covariant divergence of Eqs. (4.6), and remembering that the  $\varphi$ -equation (4.3) now contains a mass term, yields for the divergence relations following from (4.6) for the solutions of the field equations (4.3) – (4.5) the result:

$$\frac{a}{3}D^\mu f_{\mu\nu} - af_{\nu\mu}\kappa^\mu = 0 . \quad (4.16)$$

These relations are trivially satisfied for  $D_\nu\Phi^2 = 0$  implying  $f_{\mu\nu} = 0$ ; i.e. for  $\kappa_\nu = -\partial_\nu \log \Phi^2$  being “pure gauge” (compare the discussion at the end of Sect. II concerning the square of the Dirac operator). Hence Eqs. (4.16) are satisfied in the limit of the Weyl space  $W_4$  being equivalent to a Riemannian space  $V_4$ . Using now the field equations (4.5) for vanishing  $f$ -curvature, together with Eq. (4.8) and  $D_\mu\Phi^2 = 0$ , one easily concludes that

$$\kappa_\rho = 0 , \text{ i.e. that } \Phi^2 = \text{const} , \quad (4.17)$$

implying that the Weyl space  $W_4$ , in fact, reduces completely to a Riemannian space  $V_4$ .

Considering, finally, the square of the Dirac operator for the case (4.17) [compare Eqs. (2.44) and (3.29)] one sees that it is a diagonal operator with constant eigenvalue  $\tilde{\gamma}^2\Phi^2$ .

Summarizing we can say that with  $a \neq 0$  and the Weyl-symmetry breaking relation  $D_\mu\Phi^2 = 0$ , with  $\Phi^2 = \text{const}$ , we have to view Eqs. (4.3) – (4.8) as a set of covariant field equations formulated in a  $V_4$  with a definite metric (see Sect. V below). Both the  $\varphi$ -field and the  $\psi$ -field have now acquired a mass in this quasi classical (i.e. single-particle) theory

with the mass of the spinor field being determined by the modulus  $\Phi$  of the scalar field. This result is reminiscent of the situation occurring in the Higgs phenomenon of a nonabelian gauge theory (for example the standard model) where the masses of the (nonabelian) gauge and fermion fields are determined by the vacuum expectation value of the scalar field possessing a nonlinear selfcoupling of the same type as discussed here, i.e. by the *classical part* contained in the quantized scalar field of the model. Since the gauge sector – after Weyl symmetry breaking – contains in the present case only the massless  $A_\mu$ -fields [the  $U(1)$  gauge fields] and classical gravitation [the Lorentz gauge fields in a vierbein formulation] the Higgs-type mass giving phenomenon only manifests itself, besides for the  $\varphi$ -field, in the mass term for the  $\psi$ -field, obtained for constant  $\Phi^2$ , implying in turn the reduction of the Weyl space to a Riemannian space.

## V. DETERMINATION OF THE FREE PARAMETERS

Having broken in the last section the original massless Weyl and  $U(1)$  symmetric formulation of the theory to a  $U(1)$  gauge theory of interacting massive, single-particle, scalar and spinor fields formulated in a pseudo-Riemannian space  $V_4$ , i.e. taking gravitational effect into account, we now have to determine the free parameters of the theory and relate the description to the standard one and, in particular, see how Einstein's equations for the metric appear on the scene.

Considering Eqs. (4.3) - (4.8), as discussed for the case  $D_\rho\Phi^2 = 0$  with  $\Phi^2 = \text{const} > 0$ , and hence  $\kappa_\rho = 0$ , yields from (4.8)

$$R = \bar{R} = -6 \left[ \frac{mc}{\hbar} \right]^2 \Phi^2. \quad (5.1)$$

Thus the underlying space  $V_4$  is of constant negative curvature being isomorphic to the noncompact coset space [18]  $G/H = O(1, 4)/O(1, 3)$ . Furthermore, with  $\kappa_\rho = 0$  Eq. (4.5) reads  $\partial_\rho \bar{R} = 0$  in accord with (5.1).

The field equations for  $\varphi = \Phi e^{\frac{i}{\hbar} S}$  (with constant modulus) and  $\psi$  now read:

$$g^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\varphi + \left\{ \left[ \frac{mc}{\hbar} \right]^2 (a - \Phi^2) + 4\beta\Phi^2 - \tilde{\gamma} \frac{1}{\sqrt{\Phi^2}} (\bar{\psi}\psi) \right\} \varphi = 0, \quad (5.2)$$

$$-i\gamma^\mu\tilde{\nabla}_\mu\psi - \tilde{\gamma}\sqrt{\Phi^2}\psi = 0, \quad (5.3)$$

where we denote by  $\tilde{\nabla}_\mu\varphi$  and  $\tilde{\nabla}_\mu\psi$  the metric and  $U(1)$  covariant derivatives (3.4) and (3.6) for  $\kappa_\rho = 0$ , respectively.

Regarding the field equations (4.6) in the present case, we have to remember that the term proportional to  $\tilde{\alpha}$  was introduced in the Lagrangeans (2.2) and (3.8) above in order to yield a nontrivial dynamics for the Weyl vector field  $\kappa_\rho$ . Since the Weyl vector field has disappeared after the symmetry breaking  $W_4 \longrightarrow V_4$ , considering, moreover, constant  $\Phi$ , we may now take  $\tilde{\alpha} = 0$  and rewrite (4.6) as

$$\bar{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\bar{R} = \frac{1}{\frac{1}{6}[\Phi^2 + a]K} \left\{ \tilde{\Theta}_{\mu\nu}^{(\varphi)\prime} + \tilde{T}_{\mu\nu}^{(\psi)\prime} + T_{\mu\nu}^{(F)\prime} \right\}. \quad (5.4)$$

The primed tensors on the rhs of (5.4) are the energy-momentum tensors for the *massiv*  $\varphi$  and  $\psi$  fields given – together with  $T_{\mu\nu}^{(F)\prime}$  – by

$$\tilde{\Theta}_{\mu\nu}^{(\varphi)\prime} = K \left[ \tilde{\Theta}_{\mu\nu}^{(\varphi)}|_{\kappa_\rho=0} + g_{\mu\nu} \frac{a}{2} \left[ \frac{mc}{\hbar} \right]^2 \Phi^2 \right], \quad (5.5)$$

$$\tilde{T}_{\mu\nu}^{(\psi)\prime} = K \left[ \tilde{T}_{\mu\nu}^{(\psi)}|_{\kappa_\rho=0} + g_{\mu\nu} \frac{mc}{\hbar} \sqrt{\Phi^2} (\bar{\psi}\psi) \right], \quad (5.6)$$

$$T_{\mu\nu}^{(F)\prime} = K T_{\mu\nu}^{(F)}, \quad (5.7)$$

where the squared bare gravitational mass of the  $\varphi$ -field is  $a^2m^2$ , and in the mass term of the  $\psi$ -field we have taken

$$-\tilde{\gamma}\sqrt{\Phi^2} = \frac{mc}{\hbar}\sqrt{\Phi^2} = \frac{Mc}{\hbar}. \quad (5.8)$$

In accordance with (5.3) we have here adopted for the value of the constant  $\tilde{\gamma}$  of dimension  $[L^{-1}]$  characterizing the Yukawa coupling the value  $\tilde{\gamma} = -mc/\hbar$  yielding a fermion mass of the correct sign and of the size  $M = m\sqrt{\Phi^2}$  in Eq. (5.3).

In order to fix a conventional mass term in the dynamics of the  $\varphi$ -field we choose  $a = 1$  thereby introducing the length scale

$$l_\varphi = \frac{\hbar}{mc} \quad (5.9)$$

into the theory after symmetry breaking. We shall use this length scale as the intrinsic unit for measuring the constants appearing in the Lagrangean (4.1) and in the field equations derived from it. Thus Eq. (5.8) implies that  $\tilde{\gamma} = -1 \cdot l_\varphi^{-1}$ . Similarly we measure the nonlinear coupling constant  $\beta$  in units of  $l_\varphi^{-2}$  and write  $\beta = \beta' l_\varphi^{-2}$ . The conversion between mass and length in Eq. (5.9) is done by assuming  $\hbar$  and  $c$  to be given constants of nature characterizing quantum mechanics and special relativity, respectively.

Now the effective mass squared in units of  $l_\varphi^{-2}$  of the interacting  $\varphi$ -field is represented by the first three (constant) terms in the curly brackets in Eq. (5.2) given, with  $a = 1$  and (5.9), by

$$V(\Phi^2) = \left[ \frac{mc}{\hbar} \right]^2 (a - \Phi^2) + 4\beta\Phi^2 = l_\varphi^{-2} [1 - \Phi^2 + 4\beta'\Phi^2]. \quad (5.10)$$

This expression is different from the mass contribution to the energy-momentum tensor of the  $\varphi$ -field which is given by the second term on the rhs of (5.5) for  $a = 1$  accounting for the change in the energy-momentum tensor  $\tilde{\Theta}_{\mu\nu}^{(\varphi)}$  due to the breaking of the Weyl symmetry induced by  $\mathcal{L}_B$ . Of course, there also contributes a term proportional to the nonlinear coupling constant  $\beta$  (assumed to be positive) given by  $g_{\mu\nu}\beta(\Phi^2)^2$  [compare the last term in (3.17)] which is contained in  $\tilde{\Theta}_{\mu\nu}^{(\varphi)}|_{\kappa_\rho=0}$  on the rhs of (5.5) surviving in the limit  $D_\rho\Phi^2 = \partial_\rho\Phi^2 = 0$ , and appearing thus as a source term on the rhs of Eq. (5.4). Finally, the constant  $K$  in Eqs. (5.4) and (5.5) – (5.7) assures that all the source terms in the field equations (5.4), i.e. the primed energy-momentum tensors in the curly brackets, possess the dimension energy/volume [compare the discussion in Sect. II]. For electromagnetism this implies only a trivial change as given in (5.7) [see the discussion after Eqs. (3.8) and (3.19) above].

It is easy to show using Eqs. (3.24), (3.25) and (3.27) as well as the symmetry breaking relation  $D_\rho\Phi^2 = 0$ , with  $\Phi^2 = \text{const}$ , that the rhs of (5.4) obeys

$$\bar{\nabla} [\tilde{\Theta}_{\mu\nu}^{(\varphi)\prime} + \tilde{T}_{\mu\nu}^{(\psi)\prime} + \tilde{T}_{\mu\nu}^{(F)\prime}] = 0 \quad (5.11)$$

expressing the covariant energy-momentum conservation in a  $V_4$  with sources provided by the massive, charged, interacting fields  $\varphi$  and  $\psi$  together with electromagnetism.

The equations (5.4) are identical with Einstein's field equations for the metric in general relativity coupled to an interacting system of massive scalar, massive Dirac-spinor and massless electromagnetic fields provided we can identify the factor in front of the curly brackets with Einstein's gravitational constant  $\kappa_E = 8\pi N/c^4 = 2.076 \cdot 10^{-48} g^{-1} cm^{-1} sec^2$ , where  $N$  is Newton's constant, i.e. provided

$$\kappa_E = \frac{1}{\frac{1}{6} [\Phi^2 + 1] K} > 0. \quad (5.12)$$

It appears that the over all size of the gravitational coupling constant (5.12) is determined by the constant  $K$  while  $\Phi^2$  is a constant of order unity. To see what the contribution of  $\Phi^2$  relative to the choice  $a = 1$  may be let us assume for the moment that there is no fermion field present and that for a coordinate independent solution of (5.2) to exist one would have to require the vanishing of  $V(\Phi^2)$  as defined in (5.10). This would yield

$$a = 1 = (1 - 4\beta') \Phi^2. \quad (5.13)$$

relating thus  $\Phi^2$  to the value of the constant  $\beta$  measured in units of  $l_\varphi^{-2}$ . For the constant  $\kappa_E$  this would imply that in the case of a universal coordinate independent solution of the  $\varphi$ -field equation one would obtain for Einstein's gravitational constant the result

$$\kappa_E = \frac{1}{\frac{1}{6} \left[ \frac{2-4\beta'}{1-4\beta'} \right] K}. \quad (5.14)$$

This expression remains positive, for positive  $K$ , for any nonlinear universal coupling constant  $\beta' > 0$ . We thus obtain in this theory a universal gravitational coupling constant of the type (5.12) or (5.14) as in a Brans-Dicke-like scalar-tensor theory. In the following we shall choose (5.12) as the identification of the gravitational constant in the present theory which is an expression independent of the value for the effective mass of the  $\varphi$ -field obtained from (5.2).

Moreover, we have to remember that Eqs. (5.4), with the specfield source terms indicated, should yield a  $V_4$  of constant curvature according to Eqs. (5.1) and (4.5) [the latter for  $\kappa_\rho = 0$ ]. Computing the trace of (5.4) one convinces oneself with the help of (4.9) and (4.10), using also (5.1) again, that this is consistent with the field equations for the metric.

So far we have considered a closed system of interacting massive  $\varphi$ ,  $\psi$  and (massless) electromagnetic fields in interaction with the metric of the underlying space  $V_4$  which turned out to be of constant curvature. Having obtained Einstein's field equations (5.4) with the gravitational coupling constant defined by (5.12), we could now extend the description and formulate the theory in the presence of a classical background gravitational field – say of cosmological origin – by adding another (classical) source term in the curly brackets on the rhs of (5.4) and drop the requirement that the underlying space be of constant curvature.

Another extension of the theory investigated in this paper would be the extension to a multi-particle theory, in particular, by considering second quantized spinor fields in this broken Weyl theory. We shall make some remarks in this direction in Sect. VI below. However, it appears that the complex scalar field  $\varphi$  plays a special role in this formalism with its modulus remaining a classical field. It is, therefore, doubtful whether a fully quantized scalar field would lead in this context to an improved formulation of this theory.

We close this section by remarking that the essential parameter appearing after the breaking of the Weyl symmetry by  $\mathcal{L}_B$  defined in (4.2) is, with  $a = 1$ , the mass  $m$  of the scalar field, or rather the length  $l_\varphi$  defined in (5.9). This length scale was used above also as the intrinsic unit in which the dimensional constants  $\tilde{\gamma}$  and  $\beta$  appearing already in the Weyl-symmetric theory are ultimately to be measured. In order to fix the length scale  $l_\varphi$  to a particular value we could now arbitrarily identify  $m$  with the mass of the  $\pi^0$ -meson marking the lower edge of the hadronic mass spectrum. The value of the constant  $\Phi^2$ , finally, determines, according to Eq. (5.8), the deviation of the mass  $M$  of the Dirac field from the mass of the scalar field.

## VI. DISCUSSION

We have used in this paper a Weyl geometry to formulate the dynamics of a massless scalar and a massless Dirac spinor field in the presence of electromagnetic and metric fields with all the field quantities – possessing nontrivial Weyl weights – being determined up to

conformal rescalings, (1.2) and (A2), coupled to the corresponding transformations (1.3) of the Weyl vector fields  $\kappa_\rho$ . Then we explicitly broke this Weyl-invariant theory defined by the Lagrangean  $\tilde{\mathcal{L}}_{W_4}$  by adding a term  $\mathcal{L}_B$  constructed with the help of the curvature scalar of the  $W_4$  and a mass term for the  $\varphi$ -field in order to study the appearance of a mass term for the fermion field and a corresponding length scale characterizing the dynamics after breaking the symmetry of the embedding space from a Weyl space  $W_4$  to a pseudo-Riemannian space  $V_4$ .

The appearance of Einstein's equations with sources given by the energy-momentum tensors of the now massive scalar and spinor fields, together with electromagnetic fields, was investigated and a gravitational coupling constant was identified by the expression (5.12). The symmetry reduction from a Weyl space to a Riemannian space was governed by the equations  $D_\rho \Phi^2 = 0$ , which was deduced from the requirement of the square of the Dirac operator to be a diagonal operator in spin space in the massive case, and from the demand to satisfy the divergence relations following from the field equations for the metric together with the contracted Bianchi identities for a  $W_4$ . This implies the relations  $\kappa_\rho = -\partial_\rho \log \Phi^2$ , i.e. the Weyl vector field is of “pure gauge type” possessing zero length curvature,  $f_{\mu\nu} = 0$ , determining thereby the  $D(1)$ -part of the connection on the Weyl frame bundle and, in fact, reducing this bundle completely to a bundle over a  $V_4$  for a *constant* modulus  $\Phi$  as a consequence of the field equations for  $\kappa_\rho$  in the broken theory.

It is clear from the role the modulus of the scalar field plays in this theory by acting as a potential for the Weyl vector field – which itself is part of the original Weyl geometry – that the scalar field with nonlinear selfcoupling is not a true matter field describing scalar particles. It is a universal field necessary to establish a scale of length in a theory and should probably not be interpreted as a field having a particle interpretation. This raises the question whether it was reasonable to endow this field with a charge, denoted by  $q$ , and couple it to the electromagnetic fields. The question whether this charge has to be zero or is nonvanishing can not be decided in the present theory. This topic will be taken up again in connection with the study of an additional  $SU(2)$  symmetry (weak isospin) and a

corresponding representation character of  $\varphi$  with respect to this group. The extension of the group structure from  $U(1)$  to  $U(1) \otimes SU(2)$  will be addressed in a separate paper devoted to an investigation of a Weyl-invariant theory and its breaking for a case which is closest to the situation realized in the unified electroweak theory.

The result  $\Phi = \text{const} > 0$  yielding a formulation of the theory in a Riemannian space  $V_4$  leads, for the fermion field  $\psi$ , to the mass value  $M = \Phi m$  relative to the scale set by the scalar field after Weyl-symmetry breaking. The modulus  $\Phi$  of the scalar field, however, also enters the expression for the gravitational constant having a Brans-Dicke-like form as discussed in Sect. V (compare Eq. (5.4) for  $a = 1$ ). It appears that this mass shift for the  $\psi$ -field cannot yield arbitrary large values for  $M$  since, at the same time, this would reduce the gravitational coupling constant  $\kappa_E$  of the theory correspondingly. It was always a surprising feature of the conventional Higgs mechanism that masses could be shifted to arbitrary large values in the usual theory of spontaneous symmetry breaking triggered by a nonvanishing vacuum expectation value of the scalar field. No limitations for the size of the actual mass value seem to arise in the usual formalism which would, in fact, be expected to appear if gravitation were included in the theory. In the broken Weyl theory studied in this paper gravitation is present from the outset, and it is seen from the broken theory formulated in a  $V_4$  that the capacity of fermionic matter to generate gravitational fields is *diminished* when the value of  $M$  becomes very large due to the modulus  $\Phi$  becoming very large compared to unity in Eq. (5.12). Thus the mass giving phenomenon due to  $D(1)$  symmetry breaking does have an effect on gravity when studied in this Weyl-geometric framework although it is still not possible to determine the actual value of  $\Phi$ .

A further question regards the quantum nature of the scalar and spinor fields involved. So far, the theory presented in this paper is formulated in a quasi classical, single-particle description for the  $\varphi$ - and  $\psi$ -fields with the modulus  $\Phi$  of  $\varphi$  being, indeed, a classical field corresponding, as mentioned, to the vacuum expectation value of the scalar field in the conventional formulation of spontaneous symmetry breaking in gauge theories. Although it does not seem to make much sense to give to the scalar field in the present theory a fully

quantized many-particle (i.e. second quantized) form, it may nevertheless be attractive to extend the description of the fermion matter field to a many-particle formalism by introducing Fock space fibers in an associated bundle defined in analogy to the bundle (2.8). It may also be interesting to relate such a many-particle formalism to Prugovečki's programme of stochastic quantization and use the arbitrary spin fields introduced and discussed in [11] in order to extend the presented Weyl-geometric description to a theory involving geometro-stochastically quantized matter fields  $\Psi$  of arbitrary spin.

## APPENDIX A: THE GEOMETRY OF A WEYL SPACE $W_4$

A natural basis in the local tangent space  $T_x(W_4)$  at  $x \in W_4$  will be denoted by  $\partial_\mu$ ;  $\mu = 0,1,2,3$ , and, similarly, a natural basis in the dual tangent space  $T_x^*(W_4)$  at  $x \in W_4$  will be denoted by  $dx^\mu$  with the corresponding contravariant and covariant tensor components labelled with Greek indices. A local Lorentzian basis and cobasis is denoted by  $e_i$  and  $\theta^i$ ;  $i = 0,1,2,3$ , respectively, and the corresponding tensor components referring to an orthonormal frame are labelled by Latin indices. Greek indices are lowered and raised with  $g_{\mu\nu}$  and  $g^{\mu\nu}$ , respectively, while local Lorentzian indices are lowered and raised by the metric tensor of Minkowski space  $\eta_{ik}$  and  $\eta^{ik}$ , respectively. The summation convention for twice appearing Greek or Latin indices is always assumed.

Historically a connection in a  $W_4$  is given in a natural basis by the following connection coefficients:

$$\Gamma_{\mu\nu}{}^\rho = \frac{1}{2} g^{\rho\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) - \frac{1}{2} (\kappa_\mu \delta_\nu^\rho + \kappa_\nu \delta_\mu^\rho - \kappa^\rho g_{\mu\nu}) = \bar{\Gamma}_{\mu\nu}{}^\rho + W_{\mu\nu}{}^\rho \quad (\text{A1})$$

expressed in terms of the Christoffel symbols,  $\bar{\Gamma}_{\mu\nu}{}^\rho = \{\rho_{\mu\nu}\}$  determined by the metric  $g_{\mu\nu}$ , and by a Weyl addition, called  $W_{\mu\nu}{}^\rho$ , determined by the Weyl vector field  $\kappa_\rho$ . Here and in the following discussion we denote purely metric quantities pertaining to a  $V_4$  by a bar. The fourty coefficients  $\Gamma_{\mu\nu}{}^\rho$  defined by (A1) are symmetric in  $\mu$  and  $\nu$ , i.e.  $\Gamma_{\mu\nu}{}^\rho = \Gamma_{\nu\mu}{}^\rho$ . Moreover, they are invariant under Weyl transformations (1.2), (1.3) allowing thus the definition of a covariant derivative without specifying a particular metric and Weyl vector field in the class (1.1).

A tensor field  $\phi^{(n,m)}(x)$  of type  $(n, m)$ , i.e. being covariant of degree  $n$  and contravariant of degree  $m$ , has Weyl weight  $w(\phi^{(n,m)})$  if it transforms under Weyl transformations (1.2), (1.3) as

$$\phi^{(n,m)\prime}(x) = [\sigma(x)]^{w(\phi^{(n,m)})} \phi^{(n,m)}(x). \quad (\text{A2})$$

The Weyl-covariant derivative of  $\phi^{(n,m)}$ , i.e. the covariant derivative  $D = dx^\rho D_\rho$  invariant

under (1.2), (1.3) as well as under changes of the atlas on  $W_4$ , is given by (we suppress the arguments  $(x)$ )

$$D\phi^{(n,m)} = \nabla\phi^{(n,m)} - w(\phi^{(n,m)}) \kappa \phi^{(n,m)}, \quad (\text{A3})$$

where  $\nabla = dx^\rho \nabla_\rho$  with  $\nabla_\rho$  denoting the covariant derivative with respect to  $\Gamma_{\mu\nu}^\rho$  in the direction  $\partial_\rho$ . Clearly,  $D\phi^{(n,m)}$  transforms under Weyl transformations in the same manner as  $\phi^{(n,m)}$  does.

Eq. (1.2) shows that the covariant metric tensor has Weyl weight  $w(g_{\mu\nu}) = 1$  implying that

$$D_\rho g_{\mu\nu} = 0 \quad \text{is equivalent to} \quad \nabla_\rho g_{\mu\nu} = \kappa_\rho g_{\mu\nu}. \quad (\text{A4})$$

Eq. (A4) is a Weyl-covariant statement which, for  $\kappa_\rho = 0$ , goes over into the relation  $\bar{\nabla}_\rho g_{\mu\nu} = 0$  known from Riemannian geometry with  $\bar{\nabla}_\rho$  denoting the metric covariant derivative with respect to  $\bar{\Gamma}_{\mu\nu}^\rho$ . Analogously, the contravariant metric tensor  $g^{\mu\nu}$  has Weyl weight  $w(g^{\mu\nu}) = -1$ , with the determinant  $g$  of  $g_{\mu\nu}$  having Weyl weight  $w(g) = 4$ .

Let us now consider local Lorentzian frames and coframes defined by

$$e_i = \lambda_i^\mu(x) \partial_\mu; \quad \theta^i = \lambda_\mu^i(x) dx^\mu, \quad (\text{A5})$$

where  $\lambda_\mu^i(x)$  and  $\lambda_i^\mu(x)$  are the vierbein fields and their inverse, respectively, and characterize the geometry of a  $W_4$  by structural equations of Cartan type for the connection on the Weyl frame bundle

$$P_W(W_4, G = SO(3, 1) \otimes D(1)) \quad (\text{A6})$$

over a base  $W_4$  with the structural group  $G$  being the direct product of the orthochronous Lorentz group  $SO(3, 1) \equiv O(3, 1)^{++}$  and the dilatation group  $D(1)$ . (In order to make this paper selfcontained we repeat here some of the formulae appearing already in Ref. [12].)

The bundle  $P_W$  can be thought of as the reduction of the generalized linear frame bundle with structural group  $Gl(4, R)$  due to the introduction of a metric with signature  $(+, -, -, -)$

on the space-time base manifold with  $g_{\mu\nu}$  taking values in the coset space  $Gl(4, R)/SO(3, 1)$ . In  $P_W$  the metric is, however, only fixed modulo Weyl transformations. Moreover, the connection on the general linear frame bundle reduces to a connection on  $P_W$  if the metric is Weyl-covariant constant, i.e. satisfies (A4) (compare Ref. [13]).

We first observe that the vierbein fields  $\lambda_\mu^i(x)$  have Weyl weight  $w(\lambda_\mu^i) = \frac{1}{2}$ , while their inverse,  $\lambda_i^\mu(x)$ , have Weyl weight  $w(\lambda_i^\mu) = -\frac{1}{2}$ . The same applies to  $\theta^i$  and  $e_i$ , respectively. As usual the relation between metric and vierbein fields is given by

$$g_{\mu\nu}(x) = \lambda_\mu^i(x)\lambda_\nu^k(x) \eta_{ik} \quad (\text{A7})$$

with  $\eta_{ik} = \text{diag}(1, -1, -1, -1)$  assumed to have Weyl weight zero. The above convention adopted for the vierbein fields and their inverse implies that changing a Greek tensor index of a quantity into a Latin (local Lorentzian) index changes the Weyl weight of the components by half a unit.

The pull back of a connection on  $P_W$  is given by a set of one-forms  $(w_{ik}, \kappa)$ , obeying

$$\omega_{ik} = -\omega_{ki} = \theta^j \Gamma_{jik}; \quad \kappa = \theta^j \kappa_j, \quad (\text{A8})$$

where  $\Gamma_{jik}$  and  $\kappa_j$  are the connection coefficients for the Lorentz part and the  $D(1)$  part, respectively.

Cartan's formula for the relation between the  $\bar{\Gamma}_{\mu\nu}^\rho$  and the Ricci rotation coefficients  $\bar{\Gamma}_{jik}$  of a  $V_4$  – neglecting torsion in the context of this paper, which was part of Cartan's original argument in [14] – may now be extended to include the Weyl vector field  $\kappa_i = \lambda_i^\mu \kappa_\mu$  and can be written in Weyl-covariant form as

$$D\lambda_i^\mu \equiv \widehat{\nabla}\lambda_i^\mu + \frac{1}{2}\kappa\lambda_i^\mu - \omega_i{}^j\lambda_j^\mu = 0, \quad (\text{A9})$$

where  $\widehat{\nabla}\lambda_i^\mu$  denotes the covariant derivative of only the Greek index of  $\lambda_i^\mu$  with respect to  $\Gamma_{\mu\nu}^\rho$  as defined in (A1). Equation (A9) is consistent with  $Dg_{\mu\nu} = 0$  and equation (A7) as well as with the antisymmetry of the forms  $\omega_{ik}$  expressed in (A8). We mention in passing that  $\kappa_\mu$  has Weyl weight zero; correspondingly,  $\kappa_i$  has Weyl weight  $-\frac{1}{2}$ , and  $\kappa = \kappa_\mu dx^\mu = \kappa_i \theta^i$

is a one-form of Weyl weight zero. Equation (A9) now yields for the Lorentz part of the connection on  $P_W$  the result

$$\omega_{ik} = \bar{\omega}_{ik} - \frac{1}{2}(\kappa_i \theta_k - \kappa_k \theta_i), \quad (\text{A10})$$

where  $\bar{\omega}_{ik} = \theta^j \bar{\Gamma}_{jik}$  is the metric part with  $\bar{\Gamma}_{jik} = -\bar{\Gamma}_{jki}$  denoting the Ricci rotation coefficients of a  $V_4$ . It is easy to show using (A1) and (A9) that the  $\omega_{ik}$  are invariant under Weyl transformations (1.2), (1.3) obeying  $\omega'_{ik} = \omega_{ik}$ .

Remembering that the Weyl weight of  $\theta^k$  is  $\frac{1}{2}$  the structural equations of Cartan type characterizing the geometry of a  $W_4$  may now be written in Weyl-covariant form as:

$$D\theta^k \equiv d\theta^k + \omega_j^k \wedge \theta^j - \frac{1}{2}\kappa \wedge \theta^k = 0, \quad (\text{A11})$$

$$d\omega_{ik} + \omega_{ij} \wedge \omega_k^j = \Omega_{ik}, \quad (\text{A12})$$

$$d\kappa = f. \quad (\text{A13})$$

Here Eq. (A11) states that the torsion vanishes in a  $W_4$ , and Eqs. (A12) and (A13) define the curvature two-forms  $(\Omega_{ik}, f)$  on  $P_W$  (pulled back to the base) of the connection one-forms  $(\omega_{ik}, \kappa)$  given by (A8) and (A10) with  $\Omega_{ik} = -\Omega_{ki}$ .

The Bianchi identities for a  $W_4$  follow in the usual way as integrability conditions from the structural equations (A11) – (A13) by exterior derivation. They read:

$$[\Omega_{jk} - \frac{1}{2}f\eta_{jk}] \wedge \theta^j = 0, \quad (\text{A14})$$

$$D\Omega_{ij} = d\Omega_{ij} - \omega_i^k \wedge \Omega_{kj} - \omega_j^k \wedge \Omega_{ik} = 0. \quad (\text{A15})$$

If torsion is nonzero, i.e. for a Cartan-Weyl space  $CW_4$  (compare Ref. [12]) the rhs of Eq. (A14) would be given by  $D\Omega_k$ , where  $\Omega_k$  is the vector valued torsion two-form.

In parenthesis we would like to remark that one could also introduce for the characterization of a  $W_4$  a connection form

$$\hat{\omega}_{ik} = \omega_{ik} - \frac{1}{2}\kappa\eta_{ik} \quad (\text{A16})$$

which is no longer Lorentz Lie algebra valued (i.e. antisymmetric in  $i, k$ ) and write Eq. (A11) as

$$D\theta^k \equiv d\theta^k + \hat{\omega}_j{}^k \wedge \theta^j = 0. \quad (\text{A17})$$

Rewriting the structural equations (A12) and (A13) in terms of  $\hat{\omega}_{ik}$  yields

$$d\hat{\omega}_{ik} + \hat{\omega}_{ij} \wedge \hat{\omega}_k{}^j = \hat{\Omega}_{ik} = \Omega_{ik} - \frac{1}{2}f\eta_{ik}. \quad (\text{A18})$$

Here  $\hat{\Omega}_{ik}$  represents the total Weyl curvature, which appears already in (A14), being composed of Lorentz curvature,  $\Omega_{ik} = -\Omega_{ki}$ , and  $D(1)$  or “length curvature”  $f$ . It is often useful to introduce a curvature tensor which is not antisymmetric in the second pair of indices being thus of the type defined on the rhs of Eq. (A18). We shall do this below and in the text for the Greek indexed curvature tensor of a  $W_4$  which is “of  $\wedge$  type” although we omit the “hat” for simplicity in denoting the tensor components.

The  $W_4$ -curvature tensor is obtained from the two-forms (A12) by an expansion in a basis of two-forms:

$$\Omega_{ik} = \frac{1}{2}\theta^j \wedge \theta^l R_{jlik}. \quad (\text{A19})$$

In the following we prefer to work mainly with the tensor  $R_{\mu\nu\rho\lambda}$  defined by

$$\hat{\Omega}_{ik}\lambda_\rho^i\lambda_\lambda^k = \frac{1}{2}dx^\mu \wedge dx^\nu R_{\mu\nu\rho\lambda}, \quad (\text{A20})$$

where we suppress, as mentioned, for simplicity the hat on  $R_{\mu\nu\rho\lambda}$ .

It is easy to show that the tensor  $R_{\mu\nu\rho}{}^\sigma = R_{\mu\nu\rho\lambda}g^{\lambda\sigma}$  is invariant under Weyl transformations obeying

$$R'_{\mu\nu\rho}{}^\sigma = R_{\mu\nu\rho}{}^\sigma. \quad (\text{A21})$$

The curvature tensor  $R_{\mu\nu\rho}{}^\sigma$  of a  $W_4$  is composed of a metric part,  $\bar{R}_{\mu\nu\rho}{}^\sigma$ , and a Weyl addition denoted by  $P_{\mu\nu\rho}{}^\sigma$  which is expressed in terms of  $\kappa_\rho$  and its metric covariant derivatives (see below). Correspondingly, the contracted tensor

$$R_{\mu\rho} = R_{\mu\nu\rho}^{\phantom{\mu\nu\rho}\nu} = \bar{R}_{\mu\rho} + P_{\mu\rho} \quad (\text{A22})$$

is Weyl-invariant and decomposes into a metric part (the Ricci tensor) and a Weyl addition called  $P_{\mu\rho}$ . The curvature scalar

$$R = R_{\mu\nu}g^{\mu\nu} = \bar{R} + P \quad (\text{A23})$$

is a scalar of Weyl weight  $w(R) = -1$ .

Explicitly, the tensor  $R_{\mu\nu\rho\sigma}$  with Weyl weight 1 may be split into

$$R_{\mu\nu\rho\sigma} = \bar{R}_{\mu\nu\rho\sigma} + P_{\mu\nu\rho\sigma} \quad (\text{A24})$$

with the Riemann-Christoffel tensor given by ( $\bar{\Gamma}_{\mu\nu\sigma} = \bar{\Gamma}_{\mu\nu}^{\phantom{\mu\nu}\rho}g_{\rho\sigma}$ ):

$$\bar{R}_{\mu\nu\rho\sigma} = \partial_\mu \bar{\Gamma}_{\nu\rho\sigma} - \partial_\nu \bar{\Gamma}_{\mu\rho\sigma} + \bar{\Gamma}_{\mu\rho\lambda} \bar{\Gamma}_{\nu\sigma}^{\phantom{\nu\sigma}\lambda} - \bar{\Gamma}_{\nu\rho\lambda} \bar{\Gamma}_{\mu\sigma}^{\phantom{\mu\sigma}\lambda}, \quad (\text{A25})$$

and with the Weyl addition

$$\begin{aligned} P_{\mu\nu\rho\sigma} = & -\frac{1}{2}(g_{\mu\rho} \bar{\nabla}_\nu \kappa_\sigma + g_{\nu\sigma} \bar{\nabla}_\mu \kappa_\rho - g_{\mu\sigma} \bar{\nabla}_\nu \kappa_\rho - g_{\nu\rho} \bar{\nabla}_\mu \kappa_\sigma) \\ & -\frac{1}{4}(g_{\mu\rho} \kappa_\nu \kappa_\sigma + g_{\nu\sigma} \kappa_\mu \kappa_\rho - g_{\mu\sigma} \kappa_\nu \kappa_\rho - g_{\nu\rho} \kappa_\mu \kappa_\sigma) \\ & +\frac{1}{4}(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \kappa_\lambda \kappa^\lambda - \frac{1}{2} f_{\mu\nu} g_{\rho\sigma}. \end{aligned} \quad (\text{A26})$$

Clearly, the splitting in (A24) and in the contractions (A22) and (A23) is not Weyl-invariant. Only the sums of the two terms on the rhs of these equations has a definite covariant behaviour under Weyl transformations reproducing the tensor except for a factor  $[\sigma(x)]^w$  with the mentioned weight  $w$ . When we consider a splitting of a quantity into a metric (i.e.  $V_4$ ) part and a Weyl addition it is always understood that this split is made in a particular Weyl gauge and that it is in general not invariant against Weyl transformations.

Besides the Ricci tensor  $\bar{R}_{\mu\nu}$ , being symmetric in  $\mu, \nu$ , the Weyl addition in (A22) is, from (A26), given by

$$P_{\mu\nu} = P_{(\mu\nu)} + P_{[\mu\nu]} \quad (\text{A27})$$

with

$$P_{(\mu\nu)} = \frac{1}{2}(P_{\mu\nu} + P_{\nu\mu}) = -\frac{1}{2}(\bar{\nabla}_\mu\kappa_\nu + \bar{\nabla}_\nu\kappa_\mu) - \frac{1}{2}g_{\mu\nu}\bar{\nabla}^\rho\kappa_\rho + \frac{1}{2}g_{\mu\nu}\kappa^\rho\kappa_\rho - \frac{1}{2}\kappa_\mu\kappa_\nu, \quad (\text{A28})$$

and

$$P_{[\mu\nu]} = \frac{1}{2}(P_{\mu\nu} - P_{\nu\mu}) \equiv R_{[\mu\nu]} = -f_{\mu\nu}, \quad (\text{A29})$$

where the length curvature tensor is defined by

$$f_{\mu\nu} = \bar{\nabla}_\mu\kappa_\nu - \bar{\nabla}_\nu\kappa_\mu = \partial_\mu\kappa_\nu - \partial_\nu\kappa_\mu. \quad (\text{A30})$$

The curvature scalar (A23) of a  $W_4$  is, finally, given by

$$R = \bar{R} - 3\bar{\nabla}^\rho\kappa_\rho + \frac{3}{2}\kappa^\rho\kappa_\rho. \quad (\text{A31})$$

The tensor  $R_{\mu\nu\rho\sigma}$  defined by (A20) is antisymmetric in its first two indices and has symmetric and antisymmetric contributions regarding the last two indices. Below we shall derive from it a curvature tensor (the Bach tensor) having the same symmetries as the Riemann-Christoffel tensor in a  $V_4$ . Contracting the last two indices in (A24) with  $g^{\rho\sigma}$  yields

$$R_{\mu\nu\rho\sigma}g^{\rho\sigma} = P_{\mu\nu\rho\sigma}g^{\rho\sigma} = -2f_{\mu\nu}. \quad (\text{A32})$$

Sometimes the following formula is useful showing the lack of antisymmetry in the last two indices of the full  $W_4$  curvature tensor:

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} - f_{\mu\nu}g_{\sigma\rho}. \quad (\text{A33})$$

This formula together with (A22) and (A29) implies that  $R^\rho{}_{\mu\rho\nu} = R_{\mu\nu} + f_{\mu\nu} = R_{(\mu\nu)}$ .

The commutator of two Weyl-covariant derivatives of a contravariant vector  $a^\rho$  with Weyl weight  $w(a^\rho)$  is given by

$$[D_\mu, D_\nu] a^\rho = R_{\mu\nu\sigma}{}^\rho a^\sigma - w(a^\rho)f_{\mu\nu}a^\rho. \quad (\text{A34})$$

With the help of (A33) and the relation  $w(a_\rho) = w(a^\rho) + 1$  one immediately derives the corresponding relation for  $a_\rho = g_{\rho\sigma}a^\sigma$ :

$$[D_\mu, D_\nu] a_\rho = -R_{\mu\nu\rho}{}^\sigma a_\sigma - w(a_\rho) f_{\mu\nu} a_\rho. \quad (\text{A35})$$

Analogous formulae are obtained for tensors  $a_{\rho\lambda\dots}^{\mu\nu\dots}$  of higher rank with a curvature term appearing on the rhs for each contravariant or covariant index analogous to Eqs. (A34) and (A35), respectively.

The cyclic conditions following from (A14) for the curvature tensor  $R_{\mu\nu\rho\sigma}$  read

$$R_{\{\mu\nu\rho\}\sigma} = 0, \quad (\text{A36})$$

with  $\{\mu\nu\rho\}$  denoting the cyclic sum of the indices in the curly brackets. Finally, the Bianchi identities for the curvature tensors  $R_{\mu\nu\rho\sigma}$  and  $f_{\mu\nu}$  [compare (A15) and (A13)] read:

$$D_{\{\lambda} R_{\mu\nu\}}{}_{\rho\sigma} = 0, \quad (\text{A37})$$

$$D_{\{\lambda} f_{\mu\nu\}} = \partial_{\{\lambda} f_{\mu\nu\}} = 0. \quad (\text{A38})$$

Contraction of (A37) with  $g^{\nu\sigma}$  yields the Ricci-type identities for a  $W_4$

$$D^\rho R_{\mu\nu\sigma\rho} = -(D_\mu R_{\nu\sigma} - D_\nu R_{\mu\sigma}). \quad (\text{A39})$$

A second contraction, finally, leads to a formula which we would like to quote in a form used in the text:

$$D^\nu \left( R_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu} R \right) = \frac{1}{2} D^\nu f_{\mu\nu}. \quad (\text{A40})$$

The factor  $\frac{1}{2}$  on the rhs of (A40) is due to the fact that this equation derives from (A15) and (A19) and not from (A20) involving the curvature two-form  $\hat{\Omega}_{ik}$ , although we have used the contracted full curvature  $R_{\mu\nu}$  to formulate the result.

Bach [15] has introduced a  $W_4$  curvature tensor possessing the same symmetries as the Riemann-Christoffel tensor of a  $V_4$ :

$$S_{\mu\nu\rho\sigma} = S_{([\mu\nu][\rho\sigma])} = \frac{1}{4} \{ R_{\mu\nu\rho\sigma} - R_{\mu\nu\sigma\rho} + R_{\rho\sigma\mu\nu} - R_{\rho\sigma\nu\mu} \}. \quad (\text{A41})$$

Here we have exhibited the symmetry of the tensor by square and round brackets. Using (A24) and (A26) yields the following form for this tensor:

$$S_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{4}(g_{\mu\rho}f_{\nu\sigma} + g_{\nu\sigma}f_{\mu\rho} - g_{\mu\sigma}f_{\nu\rho} - g_{\nu\rho}f_{\mu\sigma}) + \frac{1}{2}f_{\mu\nu}g_{\rho\sigma}. \quad (\text{A42})$$

It is easy to show using (A36) that it, moreover, satisfies the cyclic conditions

$$S_{\{\mu\nu\rho\}\sigma} = 0. \quad (\text{A43})$$

The first contraction of (A42) yields the symmetric tensor of Weyl weight zero [compare Eqs. (A22) and (A27) – (A29)]:

$$S_{\mu\rho} = S_{\mu\nu\rho\sigma}g^{\nu\sigma} = R_{\mu\rho} + f_{\mu\rho} = R_{(\mu\rho)} = \bar{R}_{\mu\rho} + P_{(\mu\rho)}, \quad (\text{A44})$$

leading to

$$S = S_{\mu\rho}g^{\mu\rho} = R, \quad (\text{A45})$$

with  $R$  as given by (A31).

We finally write down the Weyl tensor,  $C_{\mu\nu\rho\sigma}$ , for a  $W_4$  characterized by the property of having vanishing contraction i.e.  $C_{\mu\nu\rho}{}^\nu = 0$ :

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma}) + \frac{1}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R. \quad (\text{A46})$$

This tensor may again be decomposed into a purely metric part,  $\bar{C}_{\mu\nu\rho\sigma}$ , and a  $W_4$  addition which we call  $K_{\mu\nu\rho\sigma}$ :

$$C_{\mu\nu\rho\sigma} = \bar{C}_{\mu\nu\rho\sigma} + K_{\mu\nu\rho\sigma}, \quad (\text{A47})$$

with the Weyl tensor of a  $V_4$  as given by the usual expression

$$\bar{C}_{\mu\nu\rho\sigma} = \bar{R}_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\rho}\bar{R}_{\nu\sigma} + g_{\nu\sigma}\bar{R}_{\mu\rho} - g_{\mu\sigma}\bar{R}_{\nu\rho} - g_{\nu\rho}\bar{R}_{\mu\sigma}) + \frac{1}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\bar{R}, \quad (\text{A48})$$

and the additional tensor  $K_{\mu\nu\rho\sigma}$  which is expressible in terms of the  $f_{\mu\nu}$  only:

$$K_{\mu\nu\rho\sigma} = \frac{1}{4}(g_{\mu\rho}f_{\nu\sigma} + g_{\nu\sigma}f_{\mu\rho} - g_{\mu\sigma}f_{\nu\rho} - g_{\nu\rho}f_{\mu\sigma}) - \frac{1}{2}f_{\mu\nu}g_{\rho\sigma}. \quad (\text{A49})$$

On the other hand, expressing on the rhs of (A46) the curvature tensor  $R_{\mu\nu\rho\sigma}$  and its contractions by the Bach tensor (A42) and its contractions (A44) and (A45) one at once derives the following result:

$$C_{\mu\nu\rho\sigma} = S_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\rho}S_{\nu\sigma} + g_{\nu\sigma}S_{\mu\rho} - g_{\mu\sigma}S_{\nu\rho} - g_{\nu\rho}S_{\mu\sigma}) + \frac{1}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})S + K_{\mu\nu\rho\sigma}. \quad (\text{A50})$$

This shows in comparing with (A47) that the first three terms on the rhs of this equation define a curvature tensor (possessing vanishing contraction) which is *independent* of  $\kappa_\rho$  yielding thus

$$S_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\rho}S_{\nu\sigma} + g_{\nu\sigma}S_{\mu\rho} - g_{\mu\sigma}S_{\nu\rho} - g_{\nu\rho}S_{\mu\sigma}) + \frac{1}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})S = \bar{C}_{\mu\nu\rho\sigma}. \quad (\text{A51})$$

It is well known that the tensor  $\bar{C}_{\mu\nu\rho}{}^\sigma$  is conformally invariant. Hence  $K_{\mu\nu\rho}{}^\sigma$  in (A47) is Weyl-invariant by itself since  $C_{\mu\nu\rho}{}^\sigma$  is Weyl-invariant (having Weyl weight zero). Therefore the splitting of the rhs of (A47), when the index  $\sigma$  is raised in this equation, is indeed Weyl-invariant.

Furthermore, one has

$$C_{\mu\nu\rho\sigma}g^{\rho\sigma} = C_{\mu\nu\rho}{}^\rho = K_{\mu\nu\rho}{}^\rho = -2f_{\mu\nu}. \quad (\text{A52})$$

The comparison of the right-hand sides of Eqs. (A42) and (A49), finally, shows that the tensor  $S_{\mu\nu\rho\sigma}$  possessing the symmetries indicated in Eqs. (A41) and (A43) can be expressed as

$$S_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + K_{\mu\nu\rho\sigma} + f_{\mu\nu}g_{\rho\sigma} \quad (\text{A53})$$

with an analogous decomposition of the tensor  $S_{\mu\nu\rho}{}^\sigma$  of Weyl weight zero being Weyl-invariant.

We close this appendix by mentioning the curvature invariant of Weyl weight  $-2$  constructed with the help of the tensor  $C_{\mu\nu\rho\sigma}$  (compare Bach [15]). Using (A47) and (A49) one finds

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = \bar{C}_{\mu\nu\rho\sigma}\bar{C}^{\mu\nu\rho\sigma} + \frac{3}{2}f_{\mu\nu}f^{\mu\nu}, \quad (\text{A54})$$

where the first term on the rhs may be rewritten in terms of  $S_{\mu\nu\rho\sigma}$ ,  $S_{\mu\nu}$ , and  $S$  using (A51). When multiplied with  $\sqrt{-g}$  Eq. (A54) yields a quadratic curvature invariant of Weyl weight

zero which may be considered as a possible Lagrangean density for a theory based on a  $W_4$ . The rhs of (A54), moreover, shows that this invariant is composed of an invariant expression *independent* of  $\kappa_\rho$  and an invariant constructed from the  $D(1)$ -curvature  $f_{\mu\nu}$  alone.

## APPENDIX B:

For completeness we collect in this appendix some (partly well-known) formulae used in the text involving the variation of the square root of the determinant of the metric tensor,  $g_{\mu\nu}$ , having Weyl weight  $w(\sqrt{-g}) = -2$ , and the variation of the vierbein fields under metric variations.

Varying the metric tensor one obtains the following relations:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu}, \quad (\text{B1})$$

$$g_{\mu\nu}\delta g^{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu}. \quad (\text{B2})$$

The variations of the vierbein fields are related to the variation of the metric by

$$\delta\lambda_\mu^k = -\frac{1}{2}\lambda_\sigma^k g_{\mu\rho} \delta g^{\rho\sigma}; \quad \delta\lambda_j^\nu = +\frac{1}{2}\lambda_j^\mu g_{\mu\rho} \delta g^{\rho\nu}. \quad (\text{B3})$$

This implies, for example, for the metric variation of a gradient,  $\partial_k\varphi$ , of a function  $\varphi$  expressed in a local Lorentzian frame the relation

$$\delta\partial_k\varphi = \frac{1}{2}\eta_{ki}\lambda_\rho^i\lambda_\mu^j\delta g^{\rho\mu}\partial_j\varphi. \quad (\text{B4})$$

Moreover, one finds from (A1) the formulae:

$$\Gamma_{\mu\rho}^\rho = \partial_\mu \log \sqrt{-g} - 2\kappa_\mu, \quad (\text{B5})$$

$$g^{\mu\nu}\Gamma_{\mu\nu}^\rho = -\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g} g^{\mu\rho}) + \kappa^\rho. \quad (\text{B6})$$

**APPENDIX C: THE EULER-GAUSS-BONNET AND PONTRJAGIN  
INVARIANTS FOR A  $W_4$**

There is a particular combination with Weyl weight  $-2$  composed of quadratic curvature invariants  $S_{\mu\nu\rho\sigma}S^{\mu\nu\rho\sigma}$ ,  $S_{\mu\nu}S^{\mu\nu}$ , and  $S^2$  which yields, when multiplied with  $\sqrt{-g}$ , an invariant density (a scalar of Weyl weight zero) which plays a particular role in the search for a possible Lagrangean density in a variational formulation of a theory based on a Weyl geometry. Bach [15] found that the following combination of quadratic curvature invariants leads to a contribution in a variational derivation of the field equations which is identically zero:

$$\begin{aligned} \sqrt{-g} \left\{ S_{\mu\nu\rho\sigma}S^{\mu\nu\rho\sigma} - 4S_{\mu\nu}S^{\mu\nu} + S^2 + \frac{1}{2}f_{\mu\nu}f^{\mu\nu} \right\} \\ = \sqrt{-g} \left\{ R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 + 3f_{\mu\nu}f^{\mu\nu} \right\}. \end{aligned} \quad (\text{C1})$$

To represent (C1) in a form making reference to the analogue of the Euler-Gauss-Bonnet invariant of a  $V_4$  (see Chern [16]) for the  $W_4$  case discussed here, one considers the four-form

$$\mathcal{E} = \frac{1}{32\pi^2} \varepsilon_{ijkl} \Omega^{ij} \wedge \Omega^{kl} = -\frac{1}{32\pi^2} I\eta, \quad (\text{C2})$$

where  $\Omega^{ij}$  is the curvature two-form defined in (A12), and  $\eta = \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 = \sqrt{-g}d^4x$  is the volume form. The invariant  $I$  in (C2) is given by (compare (A19); see also Ref. [17] for the case of a Riemann-Cartan space  $U_4$ ):

$$I = -\frac{1}{4} \varepsilon_{rspq} \varepsilon_{ijkl} R^{rsij} R^{pqkl} = R_{ijkl} R^{kl}{}^{ij} - 4R_{kj}R^{jk} + R^2. \quad (\text{C3})$$

One finds that the curly brackets in Eq. (C1) can be rewritten as:

$$S_{\mu\nu\rho\sigma}S^{\mu\nu\rho\sigma} - 4S_{\mu\nu}S^{\mu\nu} + S^2 + \frac{1}{2}f_{\mu\nu}f^{\mu\nu} = \underline{R}_{\mu\nu\rho\sigma}\underline{R}^{\rho\sigma\mu\nu} - 4\underline{R}_{\mu\nu}\underline{R}^{\nu\mu} + \underline{R}^2, \quad (\text{C4})$$

where we used the notation

$$\underline{R}_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2}f_{\mu\nu}g_{\rho\sigma}; \quad \underline{R}_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}f_{\mu\nu}; \quad \underline{R} = R, \quad (\text{C5})$$

relating the Greek indexed curvature tensors associated with  $\Omega^{ik}$  and  $\hat{\Omega}^{ik}$ , respectively, and its contractions [compare (A18); to avoid confusion we use an underline to denote the curvature tensor  $\underline{R}_{\mu\nu\rho\sigma} = \lambda_\mu^i \lambda_\nu^j \lambda_\rho^k \lambda_\sigma^l R_{ijkl}$  which is antisymmetric in both pairs of indices,  $\mu\nu$  and  $\rho\sigma$ ].

Thus Bach's observation that the lhs of (C1) yields identically zero when used under a variational integral implies that the form  $\mathcal{E}$  with the invariant  $I$  given by (C3) is exact, yielding upon integration in the case of a closed orientable (properly) Riemannian manifold the Euler-Poincaré characteristic of the manifold, and yielding for a Riemannian submanifold with boundary the Gauss-Bonnet theorem. As shown above, it is the Weyl-invariant density (C1) involving the curvature invariant  $I$  with the particular summation over the indices shown in (C3) and (C4) which represents, according to Bach's result, the quantity corresponding to the Euler-Gauss-Bonnet invariant for a Weyl space  $W_4$  being, moreover, independent of the particular Weyl gauge adopted in the family (1.1).

Bach also investigated two further quadratic curvature invariants involving the dual curvature tensors of Weyl weights  $w({}^*f_{\mu\nu}) = 0$  and  $w({}^*R_{\mu\nu\rho\sigma}) = 1$ :

$${}^*f_{\mu\nu} = \frac{1}{2}\sqrt{-g} \varepsilon_{\mu\nu\rho\lambda} f^{\rho\lambda}, \quad \text{and} \quad {}^*R_{\mu\nu\rho\sigma} = \frac{1}{2}\sqrt{-g} \varepsilon_{\mu\nu\delta\lambda} R^{\delta\lambda}_{\rho\sigma}. \quad (\text{C6})$$

He showed that the following invariants yield again a vanishing contribution in a variational derivation of field equations in a  $W_4$  (compare [15], Sect. XII):

$$\sqrt{-g} f_{\mu\nu} {}^*f^{\mu\nu}, \quad \text{and} \quad \sqrt{-g} R_{\mu\nu\rho\sigma} {}^*R^{\mu\nu\rho\sigma}. \quad (\text{C7})$$

The expressions (C7) may be related to the Pontrjagin invariant for a  $W_4$  which, in analogy to the Riemannian case, may be defined by the four-form of Weyl weight zero:

$$\mathcal{P} = \frac{1}{8\pi^2} \Omega_{ij} \wedge \Omega^{ij} = J \eta, \quad (\text{C8})$$

with

$$J = \frac{1}{16\pi^2} R_{ijkl} {}^*R^{ijkl} = \frac{1}{16\pi^2} \underline{R}_{\mu\nu\rho\sigma} {}^*\underline{R}^{\mu\nu\rho\sigma}, \quad (\text{C9})$$

where in the last equality in (C9) we have used the notation introduced in (C4). Bach's observation that also the invariants (C7) yield identically zero when used in a variational derivation of the field equations thus implies, together with the identity

$$\sqrt{-g} \underline{R}_{\mu\nu\rho\sigma} {}^*\underline{R}^{\mu\nu\rho\sigma} = \sqrt{-g} (R_{\mu\nu\rho\sigma} {}^*R^{\mu\nu\rho\sigma} - f_{\mu\nu} {}^*f^{\mu\nu}), \quad (\text{C10})$$

and Eq. (C9), that  $\mathcal{P}$  is an exact form on  $W_4$  which is invariant under Weyl transformations.

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